# Exercise III

**1** If the derivative of f(x) wrt x is $\frac{\cos x}{f(x)}$ then show that f(x) is a periodic function.

Sol Given $f'(x)=\frac{\cos x}{f(x)}$ $\Leftrightarrow f(x)\cdot f(x)=\cos x$

Integration both sides w.r.t.x $\left(f(x)\right)^2=\sin x+c$

$f(x)=\pm\sqrt{\sin x+c}$ where , ( $c\in$ Real constant $n=\pm 1$)

**2** Find the range of the function, $f(x)=\int_{-1}^{1}\frac{\sin x\ dt}{1-2t\cos x+t^2}$ .

**Sol.** $f(x)=\int_{-1}^{1}\frac{\sin x\ dt}{1-2t\cos x+t^2}$

$$=\sin x\int_{-1}^{1}\frac{1}{t^2-2t\cos x+\cos^2 x+1-\cos^2 x}dt$$

$$=\sin x\int_{-1}^{1}\frac{1}{(t-\cos x)^2+(\sin x)^2}dt$$

$$=\sin x\ \frac{1}{|\sin x|}\left[\tan^{-1}\left(\frac{t-\cos x}{\sin x}\right)\right]_{-1}^{1}$$

$$=\frac{\sin x}{|\sin x|}\left(\tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)-\tan^{-1}\left(\frac{-1-\cos x}{\sin x}\right)\right)$$

$$=\frac{\sin x}{|\sin x|}\left(\tan^{-1}\left(\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)+\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)\right)$$

$$=\frac{\sin x}{|\sin x|}\left(\tan^{-1}\left(\tan\frac{x}{2}\right)+\tan^{-1}\left(\cot\frac{x}{2}\right)\right)$$

$$=\frac{\sin x}{|\sin x|}\left(\tan^{-1}\left(\tan\frac{x}{2}\right)+\tan^{-1}\left(\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right)\right)=\frac{\pi}{2}+\frac{\sin x}{|\sin x|}$$

**3** A function f is defined in $[-1, 1]$ as $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$; $x \neq 0$; $f(0) = 0$; $f(1/\pi) = 0$. Discuss the continuity and derivability of f at $x = 0$.

**Sol.** $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}, x \neq 0$

$$f(x) = \int 2x\sin\frac{1}{x} - \cos\frac{1}{x}\,dx$$

$$= 2\int x\sin\frac{1}{x} - \int \cos\frac{1}{x}\,dx$$

$$= 2\left[\frac{x^2}{2}\sin\frac{1}{x} - \int\frac{x^2}{2}.\cos\frac{1}{x}.\left(\frac{-1}{x^2}\right)dx - \int\cos\frac{1}{x}\,dx\right.$$

$$= x^2\sin\frac{1}{x} + \int\cos\frac{1}{x}\,dx - \int\cos\frac{1}{x}\,dx + c$$

$$f(x) = x^2\sin\frac{1}{x} + c \qquad f\left(\frac{1}{\pi}\right) = \frac{1}{\pi^2}\sin\pi + c$$

$$c = 0$$

RHL $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0^+} x^2\sin\frac{1}{x} = 0$

LHL $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} x^2\sin\frac{1}{x} = 0$

$f(0) = 0$

as f(x) is continuous at $x = 0$

RHD $\lim\limits_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^+}\frac{x^2\sin\frac{1}{x}-0}{x-0} = \lim\limits_{x\to 0^+} x\sin\frac{1}{x} = 0$

LHD $\lim\limits_{x\to 0^-}\frac{f(x)-f(0)}{x-0}$

$$= \lim_{x\to 0^-}\frac{x^2\sin\frac{1}{x}-0}{x-0} = \lim_{x\to 0^-} x\sin\frac{1}{x} = 0$$

differentiable at $x = 0$

4 Let $f(x) = \begin{cases} -1 & \text{if } -2 \le x \le 0 \\ |x-1| & \text{if } 0 < x \le 2 \end{cases}$ and $g(x) = \int_{-2}^{x} f(t)\,dt$. Define g (x) as a function of x and test the continuity and differentiability of g(x) in (–2, 2).

**Sol.** $f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ -(x-1), & 0 < x < 1 \\ (x-1), & 1 \le x \le 2 \end{cases}$

$$f(x) = \int_{-2}^{x} f(t)dt$$

**Case I** $-2 \le x \le 0$

$$f(x) = \int_{-2}^{x} -1dt = -[t]_{-2}^{x} = -(x+2)$$

**Case II** $0 < x < 1$

$$f(x) = \int_{-2}^{0} -1dt + \int_{0}^{x} -(t+1)dt$$

$$= -(0+2) - \left(\frac{t^2}{2} - t\right)_0^x$$

$$= -2 - \left(\frac{x^2}{2} - x\right) = -2 - \frac{x^2}{x} + x$$

**Case II** $1 \le x \le 2$

$$f(x) = \int_{-2}^{\infty} -1\,dt - \int_{0}^{1}(t-1)dt + \int_{1}^{x}(t-1)dt$$

$$= -(0+2) - \left(\frac{t^2}{2} - t\right)_0^1 + \left(\frac{t^2}{2} - t\right)_1^x$$

$$= -2 - \left(\frac{1}{2} - 1\right) + \frac{x^2}{2} - x - \left(\frac{1}{2} - 1\right)$$

$$-2 + \frac{1}{2} + \frac{x^2}{2} - x + \frac{1}{2}$$

$= -1 + \frac{x^2}{2} - x$

Now $f(x) = \begin{cases} -(x+2) & -2 \le x \le 0 \\ -2 - \frac{x^2}{2} + x & 0 < x < 1 \\ -1 + \frac{x^2}{2} - x & 1 \le x \le 2 \end{cases}$

checking continous at x = 0

LHL – (0 + 2) = – 2

RHL – 2 + 0 + 0 = – 2

continuous at x = 0

checking continouty at x = 1

LHL $-2 - \frac{1}{2} + 1 = -\frac{3}{2}$

RHL $= -1 + \frac{1}{2} - 1 = -\frac{3}{2}$

continuous at x = 1

$f'(x) = \begin{cases} -1 & -2 \le x \le 0 \\ -x+1 & 0 < x < 1 \\ x-1 & 1 \le x \le 2 \end{cases}$

$f'(0^{-1}) = -1$ | $f'(1^-) = -1 + 1 = 0$

$f'(0^+) = 1$ | $f'(1^+) = 1 - 1 = 0$

Not defferentiable at x = 0 & defferentiable at x = 1 **Ans.**

**5** If $\phi(x) = \cos x - \int_0^x (x-t)\ \phi(t)\,dt$. Then find the value of $\phi''(x) + \phi(x)$.

**Sol.** $\phi(x) = \cos x - \int_0^x (x-t)\ \phi(t)\,dt$

$\phi'(x) = -\sin - \left[\int_0^x \frac{d}{dx}(x-t)\phi(t)dt + (x-x)\phi(x)\frac{d}{dx}(x) - (0-t)\phi(0)\frac{d}{dx}(0)\right]$

$= -\sin x - \int_0^x \phi(t)dt$

$f''(x) = -\cos x - \phi(x)$ so $\phi''(x) + \phi(x) = -\cos x$ **Ans.**

**6** If $y=\frac{1}{a}\int_0^x f(t)\cdot\sin a(x-t)\,dt$ then prove that $\frac{d^2y}{dx^2}+a^2y=f(x)$.

**Sol.** $y=\frac{1}{a}\int_0^x f(t)\cdot\sin a(x-t)\,dt$

$$\frac{dy}{dx}=\frac{1}{a}\left[\int_0^x \frac{d}{dx}f(t)\sin a(x-t)dt+f(x)\sin a(x-t)\frac{d}{dx}(x)-f(0)\sin a(x-0)\frac{d}{dx}(0)\right]$$

$$=\frac{1}{a}\int_0^x f(t)\cos a(x-t)(-a)\,dt$$

$$\frac{dy}{dx}=\int_0^x f(t)\cos a\,(x-t)dt$$

$$\frac{d^2y}{dx^2}=\left[\int_0^x \frac{d}{dx}f(t)\cos a(x-t)dt+f(x)\cos a(x-a)\frac{d}{dx}(x)-f(0)\cos a(x-0)\frac{d}{dx}(0)\right]$$

$$=\left[-a\int_0^x f(t)\sin a(x-t)+f(x)\right]=-a^2y+f(x)$$

$$\frac{d^2y}{dx^2}+a^2y=f(x)$$ **Ans.**

**7** If $y=x^{\int_1^x \ln t\,dt}$, find $\frac{dy}{dx}$ at $x=e$.

**Sol.** $y=x^{\int_1^x \ln t\,dt}$

$=x^{[t\log t-t]_1^x}$

$=x^{(x\log x-x+1)}$

$\log y=(x\log x-x+1)\log x$

$$\frac{1}{y}\frac{dy}{dx}=\left(\frac{x\log x-x+1}{x}\right)+(\log x)(\log x+1-1)=\frac{(x\log x-x+1)}{x}+(\log x)^2$$

putting $x=e$

$$\frac{dy}{dx}=e^{(e\log e-e+1)}\left[\frac{e\log e-e+1}{e}+(\log e)^2\right]=e\left(\frac{1}{e}+1\right)=(1+e)$$ **Ans.**

**8** A curve $C_1$ is defined by: $\frac{dy}{dx} = e^x \cos x$ for $x \in [0, 2\pi]$ and passes through the origin. Prove that the roots of the function (other than zero) occurs in the ranges $\frac{\pi}{2} < x < \pi$ and $\frac{3\pi}{2} < x < 2\pi$.

**Sol.** $\frac{dy}{dx} = e^x \cos x$ , $\int dy = \int e^x \cos x \, dx$

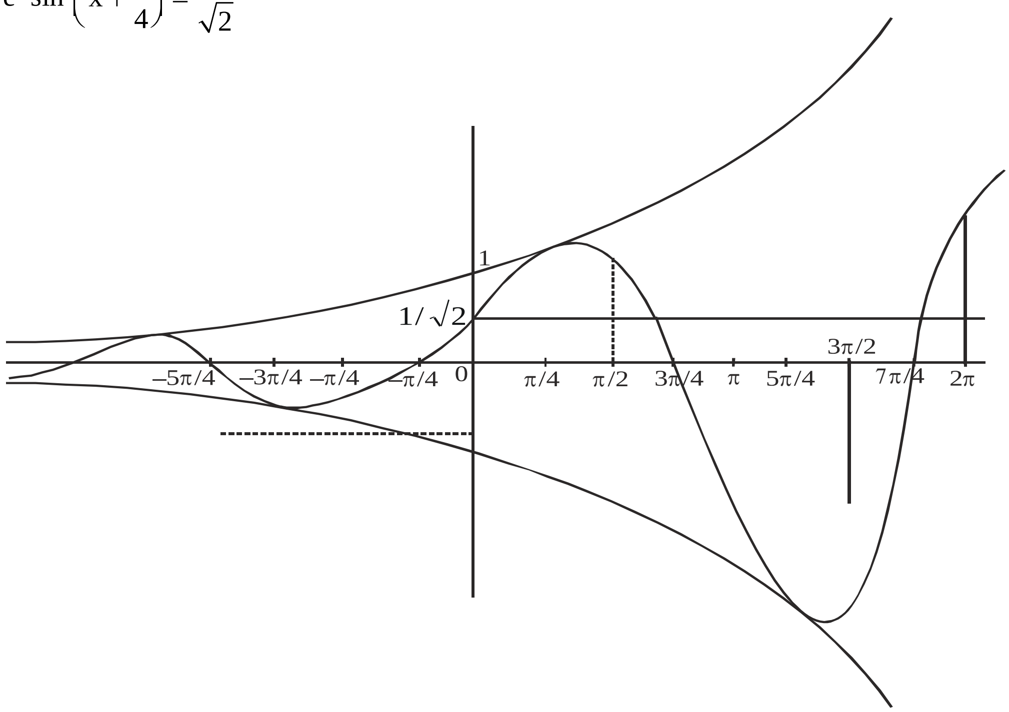
$y = \frac{e^x}{2}(\cos x + \sin x) + c$ , putting x = 0, y = 0 , $0 = \frac{e^0}{2}(\cos 0 + \sin 0) + c$

$0 = \frac{1}{2}(1) + c$ , $c = -\frac{1}{2}$ , $y = \frac{e^x}{2}(\cos x + \sin x) - \frac{1}{2}$

putting y = 0

$\frac{e^x}{2}(\cos x + \sin x) - \frac{1}{2} = 0$ , $e^x(\cos x + \sin x) = 1$

$e^x \sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$




as one root lies between

$\frac{\pi}{2}$ & $\pi$ & other lies bet $\frac{3\pi}{2}$ & $2\pi$ **Ans.**

**9**

(a) Let $g(x) = x^c . e^{2x}$ & let $f(x) = \int_0^x e^{2t} . (3t^2 + 1)^{1/2}\, dt$ . For a certain value of 'c' , the limit of $\frac{f'(x)}{g'(x)}$ as $x \to \infty$ is finite and non zero . Determine the value of 'c' and the limit .

[Sol : $g'(x) = c\, x^{c-1} . e^{2x} + x^c . e^{2x} . 2$

$f'(x) = e^{2x} (3x^2 + 1)^{1/2}$

$$\underset{x\to\infty}{\text{Limit}} \frac{f'(x)}{g'(x)} = \underset{x\to\infty}{\text{Limit}} \frac{e^{2x}\left(3x^2+1\right)^{1/2}}{c\,x^{c-1} . e^{2x} + 2x^c . e^{2x}} = \underset{x\to\infty}{\text{Limit}} \frac{x\left(3+\frac{1}{x^2}\right)^{1/2}}{x^c\left(\frac{c}{x}+2\right)}$$

If $x \to \infty$ it will be finite if $c = 1$ and $\underset{x\to\infty}{\text{Limit}}$ will be $\frac{\sqrt{3}}{2}$ ]

(b) Find the constants 'a' (a > 0) and 'b' such that , $\underset{x\to 0}{\text{Limit}} \frac{\int_0^x \frac{t^2\, dt}{\sqrt{a+t}}}{bx - \sin x} = 1$ .

[Sol : $\frac{0}{0}$ form hence using L` Hospitals rule

$$l = \underset{x\to 0}{\text{Limit}} \frac{\frac{x^2}{\sqrt{a+x}}}{b - \cos x} \qquad \text{for existence of limit } \underset{x\to 0}{\text{Limit}}\ b - \cos x = 0$$

$\Rightarrow$ b = 1

hence $\underset{x\to 0}{\text{Limit}} \frac{x^2}{1-\cos x} . \underset{x\to 0}{\text{Limit}} \frac{1}{\sqrt{a+x}} = 1 \qquad \frac{2}{\sqrt{a}} = 1$

$\Rightarrow$ a = 4 ]

**10** Evaluate: $\underset{x\to+\infty}{\text{Lim}} \frac{d}{dx} \int_{2\sin\frac{1}{x}}^{3\sqrt{x}} \frac{3t^4+1}{(t-3)(t^2+3)} dt$

[Sol. Use Leibniz's Rule. We know that

$$\frac{d}{dx}\int_{2\sin\frac{1}{x}}^{3\sqrt{x}} f(t)\,dt = f\left(3\sqrt{x}\right) D\left(3\sqrt{x}\right) - f\left(2\sin\frac{1}{x}\right) D\left(2\sin\frac{1}{x}\right)$$

$$\frac{d}{dx}\int_{2\sin\frac{1}{x}}^{3\sqrt{x}} \frac{3t^4+1}{(t-3)(t^2+3)} dt = f\left(3\sqrt{x}\right)\frac{3}{2\sqrt{x}} + f\left(2\sin\frac{1}{x}\right)\frac{2}{x^2}\cos\frac{1}{x}$$

$$= \frac{3}{2}\frac{243x^2+1}{\sqrt{x}\left(3\sqrt{x}-3\right)(9x+3)} + 2\frac{\cos\left(\frac{1}{x}\right)\left(48\sin^4\left(\frac{1}{x}\right)+1\right)}{x^2\left(2\sin\left(\frac{1}{x}\right)-3\right)\left(4\sin^2\left(\frac{1}{x}\right)+3\right)}$$

simplifying and passing to the limit (using extended real number arithmetic) we find that the second term tends to 0 and so

$$\lim_{x\to+\infty} \frac{d}{dx}\int_{2\sin\frac{1}{x}}^{3\sqrt{x}} \frac{3t^4+1}{(t-3)(t^2+3)}dt = \frac{27}{2} = 13.5 \quad \text{Ans.} ]$$

**11** If $U_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x}\, dx$, then show that $U_1, U_2, U_3, \ldots.., U_n$ constitute an AP. Hence or otherwise find the value of $U_n$.

Sol $$U_n - U_{n-1} = \int_0^{\frac{\pi}{2}} \frac{\sin^2 nx - \sin^2(n-1)x}{\sin^2 x}\cdot dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{(\sin nx + \sin(n-1)x)(\sin nx - \sin(n-1))}{\sin^2 x}$$

$$= \int_0^{\frac{\pi}{2}} \frac{2\left(\sin\left(nx - \frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\right)\left(2\sin\frac{x}{2}\cos\left(nx - \frac{x}{2}\right)\right)}{\sin^2 x}\cdot dx = \int_0^{\frac{\pi}{2}} \frac{\sin x - \sin((2n-1)x)}{\sin^2 x}\cdot dx$$

So,

$$U_n - U_{n-1} = \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{\sin x}\cdot dx = f(n),\ \text{say}$$

$$f(n) - f(n-1)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x - \sin(2n-3)x}{\sin x}\cdot dx = \int_0^{\frac{\pi}{2}} \frac{2\sin x\cos(2nx)}{\sin x}\cdot dx$$

$2nx = t \qquad\qquad \Rightarrow 2ndx = dt$

Hence,

$$f(n) - f(n-1)$$

$$= \underbrace{\int_0^{\pi} 2\cos t\left(\frac{dt}{2n}\right)}_{0}$$

Hence,

$$f(n) = f(n-1)$$

i.e. $U_n - U_{n-1}$ is a constant (Independent of n)

( $U_n - U_{n-1} = U_{n-1} - U_{n-2}$ , $\frac{U_n + U_{n-2}}{2} = U_{n-1}$ i.e. $U_n, U_{n-1}, U_{n-2}$ are in A.P )

Hence, $U_1, U_2, \ldots., U_n$ caonstitutes an A.P

$$U_1 = \int_0^{\frac{\pi}{2}} dx \left(\frac{\pi}{2}\right) \quad , \quad U_2 = \int_0^{\frac{\pi}{2}} \frac{\sin^2 2x}{\sin^2 x} \cdot dx$$

$$= 4\int_0^{\frac{\pi}{2}} \cos^2 x \cdot dx = 2\int_0^{\frac{\pi}{2}} \left(\sin^2 x + \cos^2 x\right) \cdot dx \qquad \left(\int_a^b f = \int_a^b f\left(a+b-x\right)\right)$$

$$U_2 = \pi$$

$$U_2 - U_1 = \text{(common difference of A.P)} = \left(\frac{\pi}{2}\right)$$

Hence,

$$U_n = U_1 + \left(n-1\right)\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \left(n-1\right)\frac{\pi}{2}$$

$$\boxed{U_n = n\frac{\pi}{1}}$$

**12** If $\int_0^{\infty} \frac{\ell n\, t}{x^2 + t^2}\, dt = \frac{\pi\ \ell n\, 2}{4}$ (x > 0) then show that there can be two integral values of 'x' satisfying this equation. [Ans: x = 2 or 4]

[Solution: put $t = x \tan\theta$

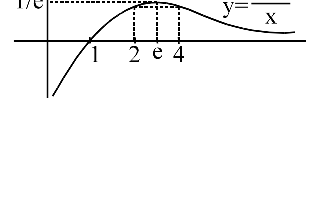
$$I = \int_0^{\pi/2} \frac{\ell n\ (x\tan\theta) \cdot x \sec^2\theta}{x^2(1+\tan^2\theta)}\, d\theta$$

$$= \frac{1}{x}\int_0^{\pi/2} (\ell n\, x + \ell n \tan\theta)\, d\theta$$

$$= \frac{\ell n\, x}{x}\int_0^{\pi/2} d\theta + \frac{1}{x}\int_0^{\pi/2} \ell n \tan\theta\, d\theta \qquad = \frac{\pi\ \ell n\ x}{2\ x} + \text{zero}$$

Hence $\frac{\pi\ \ell n\ x}{2\ x} = \frac{\pi\ \ell n\ 2}{4} \Rightarrow \frac{\ell n\ x}{x} = \frac{\ell n\ 2}{2} \Rightarrow x = 2$ or $4$

Note from the graph of $y = \frac{\ell n\ x}{x}$




that for all values of $y = \frac{\ell n\ x}{x} \in \left(0, \frac{1}{e}\right)$, there can be two values of x on either side of x = e for which $\frac{\ell n\ x}{x}$ will have the same value.]

**13** $\lim\limits_{x\to 0}\left(\int_0^1 (by+a(1-y))^x\,dy\right)^{1/x}$ (where, $b \neq a$)

[Sol. Consider $I = \int_0^1 (by+a(1-y))^x\,dy$

$$= \int_0^1 (a+(b-a)y)^x\,dy = \left[\frac{(a+(b-a)y)^{x+1}}{(x+1)}\cdot\frac{1}{b-a}\right]_0^1$$

$$I = \frac{1}{(x+1)(b-a)}\left(b^{x+1}-a^{x+1}\right) = \frac{1}{(x+1)}\left(\frac{b^{x+1}-a^{x+1}}{b-a}\right)$$

now $$L = \lim_{x\to 0}\left(\frac{b^{x+1}-a^{x+1}}{b-a}\right)^{1/x}\cdot\left(\frac{1}{(x+1)}\right)^{1/x} = \underbrace{\lim_{x\to 0}\left(\frac{1}{(x+1)}\right)^{1/x}}_{1^\infty}\cdot\underbrace{\lim_{x\to 0}\left(\frac{b^{x+1}-a^{x+1}}{b-a}\right)^{1/x}}_{1^\infty}$$

$$\left(\begin{array}{l}\lim\limits_{x\to 0}(x+1)^{\frac{1}{x}} = e^{\lim\limits_{x\to 0}\frac{1}{x}(x+1-1)} = e \\ \Rightarrow \quad \dfrac{1}{(x+1)^{1/x}} = \dfrac{1}{e}\end{array}\right)$$

$\therefore$ $$L = \frac{1}{e}\cdot\underbrace{\lim_{x\to 0}\left(\frac{b^{x+1}-a^{x+1}}{b-a}\right)^{1/x}}_{l}$$

now, $$l = e^{\lim\limits_{x\to 0}\frac{1}{x}\left(\frac{b^{x+1}-a^{x+1}-b+a}{b-a}\right)}$$

$$= e^{\frac{1}{b-a}\lim\limits_{x\to 0}\frac{b(b^x-1)-a(a^x-1)}{x}} = e^{\frac{1}{b-a}(b\ln b - a\ln a)} = e^{\ln\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}} = \left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

$\therefore$ $$L = \frac{1}{e}\cdot\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$ Ans. ]

**14** Let $a$, $b$ are real number such that $a + b = 1$ then find the minimum value of the integral $\int_0^\pi (a\sin x + b\sin 2x)^2\,dx$. [Ans. $\pi/4$]

[Sol. Let $I = \int_0^\pi (a\sin x + b\sin 2x)^2\,dx$

$$I = \int_0^{\pi} (a\sin x - b\sin 2x)^2\,dx$$

add $\quad 2I = 2\int_0^{\pi} (a^2\sin^2 x + b^2\sin^2 2x)\,dx$

$$I = 2\int_0^{\pi/2} (a^2\sin^2 x)\,dx + 2\int_0^{\pi/2} (b^2\sin^2 2x)\,dx = 2a^2\frac{\pi}{4} + 2b^2\underbrace{\int_0^{\pi/2}\sin^2 2x\,dx}_{J}$$

Let $\quad J = \int_0^{\pi/2}\sin^2 2x\,dx$ ; $\quad$ put $2x = t$

$$= \frac{1}{2}\int_0^{\pi}\sin^2 t\,dt = \int_0^{\pi/2}\sin^2 t\,dt = \frac{\pi}{4}$$

hence $\quad I = \frac{\pi a^2}{2} + \frac{\pi b^2}{2} = \frac{\pi}{2}(a^2 + b^2)$

$$I(a) = \frac{\pi}{2}[a^2 + (1-a)^2] = \frac{\pi}{2}[2a^2 - 2a + 1] = \pi\left[a^2 - a + \frac{1}{2}\right] = \pi\left[\left(a - \frac{1}{2}\right)^2 + \frac{1}{4}\right]$$

$\therefore \quad$ I(a) is minimum when $a = \frac{1}{2}$ and minimum value $= \frac{\pi}{4}$ **Ans.** ]

**15** Find a positive real valued continuously differentiable functions $f$ on the real line such that for all x

$$f^2(x) = \int_0^x \left[(f(t))^2 + (f'(t))^2\right]dt + e^2$$

[Sol. differentiating both sides w.r.t. x

$$2f(x)\cdot f'(x) = (f(x))^2 + (f'(x))^2$$

or $\quad (f(x) - f'(x))^2 = 0 \quad\Rightarrow\quad f'(x) = f(x)$

(from the given relation $f(0) = e^2 \Rightarrow f(0) = e$ or $-e$ (to be rejected) )

now $\quad \frac{f'(x)}{f(x)} = 1 \quad\Rightarrow\quad \ln(f(x)) = x + C$ ; but $f(0) = e$

$\therefore \quad \ln(e) = C \quad\Rightarrow\quad C = 1$

$\therefore \quad \ln(f(x)) = x + 1 \quad\Rightarrow\quad f(x) = e^{x+1}$ Ans. ]

**16** Let f(x) be a continuously differentiable function then prove that,

$\int_1^x [t]\,f'(t)\,dt = [x].\,f(x) - \sum_{k=1}^{[x]} f(k)$ where $[\cdot]$ denotes the greatest integer function and $x > 1$.

[Sol. $\int_1^2 f'(t)\,dt + 2\int_2^3 f'(t)\,dt + 3\int_3^4 f'(t)\,dt + \ldots\ldots + [x]\int_{[x]}^{x} f'(t)\,dt$

$= [f(t)]_1^2 + 2[f(t)]_2^3 + 3[f(t)]_3^4 + \ldots\ldots + [x]\,[f(t)]_{[x]}^{x}$

$= (f(2) - f(1)) + 2(f(3) - f(2)) + 3(f(4) - f(3)) + \ldots\ldots + [x](f(x) - f([x]))$

$= -(f(1) + f(2) + f(3) + \ldots\ldots + f([x])) + f(x).[x] = f(x).[x] - \sum_{k=1}^{[x]} f(k)$ ]

**17** Let $F(x) = \int_{-1}^{x}\sqrt{4+t^2}\,dt$ and $G(x) = \int_{x}^{1}\sqrt{4+t^2}\,dt$ then compute the value of $(FG)'(0)$ where dash denotes the derivative. [Ans. zero]

[Sol. $F(x) = \int_{-1}^{x} f(t)\,dt$ and $G(x) = \int_{x}^{1} f(t)\,dt$ where $f(t) = \sqrt{4-t^2}$

now $H(x) = F(x)\cdot G(x)$

$H'(x) = F(x)\cdot G'(x) + G(x)\cdot F'(x)$

$$H'(x) = \left(\int_{-1}^{x} f(t)\,dt\right)\left(-\sqrt{4+x^2}\right) + \left(\int_{x}^{1} f(t)\,dt\right)\left(\sqrt{4+x^2}\right)$$

$$H'(x) = \sqrt{4+x^2}\left[\int_{x}^{1}\sqrt{4+t^2}\,dt - \int_{-1}^{x}\sqrt{4+t^2}\,dt\right];\ H'(0) = 2\left[\int_{0}^{1}\sqrt{4+t^2}\,dt - \int_{-1}^{0}\sqrt{4+t^2}\,dt\right]$$

put $t = -y$

$$= \left[\int_{0}^{1}\sqrt{4+t^2}\,dt + \int_{1}^{0}\sqrt{4+y^2}\,dy\right] = \left[\int_{0}^{0}\sqrt{4+t^2}\,dt\right] = \text{zero} \quad \text{Ans. ]}$$

**19** Evaluate:

(a) $\lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\ldots\ldots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$ ;

(b) $\lim\limits_{n\to\infty}\frac{1}{n}\left[\frac{1}{n+1} + \frac{2}{n+2} + \ldots\ldots + \frac{3n}{4n}\right]$

Sol (a) Let,

$$S = \lim_{n\to\infty}\left(\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right)^{\frac{1}{n}} \qquad \because (S > 0)$$

$$\Rightarrow \log S = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r^2}{n^2}\right)$$

$$=\int_0^1 \log\left(1+x^2\right).dx$$ (Definite Integral as limit of sum.)

In tegrating by parts,

$$\Rightarrow \log S=\left(x\log\left(1+x^2\right)\right)_0^1-\int_0^1 \frac{2x^2}{1+x^2}\cdot dx$$

$$\uparrow$$
$$(\log 2)$$

$$\Rightarrow \log\left(\frac{S}{2}\right)=2\left[\int_0^1 \frac{dx}{1+x^2}-\int_0^1 .dx\right]$$

$$=2\left[\frac{\pi}{4}-1\right] \quad =\left(\frac{\pi-4}{2}\right)$$

$$\Rightarrow S=2e^{\left(\frac{\pi-4}{2}\right)}.$$

(b) Let ,

$$S=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{n+1}+\frac{2}{n+2}+\cdots+\frac{3n}{4n}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{3n}\frac{\left(\frac{r}{3n}\right)(3)}{1+\left(\frac{r}{3n}\right)(3)}$$

$$=3\int_0^1 \frac{3x}{1+3x}\cdot dx$$ (Using Inegra as limit of sum)

$$=3\int_0^1 1-\frac{1}{1+3x}\cdot dx \qquad =3\left[1-\left(\frac{In(1+3x)}{3}\right)_0^1\right]$$

$$\boxed{S=(3-In4)}.$$

**20** Let $P_n=\sqrt[n]{\frac{(3n)!}{(2n)!}}$ (n = 1, 2, 3........) then find $\lim\limits_{n\to\infty}\frac{P_n}{n}$.

**Sol.** $P_n=\left(\frac{(3n)!}{(2n)!}\right)^{1/n}=\left(\frac{(2n)!(2n+1)(2n+2)......(2n+n)}{(2n)!}\right)^{1/n}$

$$\therefore \quad \frac{P_n}{n}=\left(\frac{(2n+1)(2n+2)......(2n+n)}{n^n}\right)^{1/n}=\left(\frac{(2n+1)}{n}\cdot\frac{(2n+2)}{n}.........\frac{(2n+n)}{n}\right)^{1/n}$$

$$\therefore \quad ln\left(\frac{P_n}{n}\right)=\frac{1}{n}\sum_{r=1}^{n} ln\left(\frac{2n+r}{n}\right)=\frac{1}{n}\sum_{r=1}^{n} ln\left(2+\frac{r}{n}\right)$$

$\therefore \quad \lim_{n\to\infty} ln\left(\frac{P_n}{n}\right) = \int_0^1 ln(2+x)\,dx = x\,ln(2+x)\Big]_0^1 - \int_0^1 \frac{x}{x+2}dx$

$= ln\,3 - \left(\int_0^1 dx - \int_0^1 \frac{2\,dx}{x+2}\right) = ln\,3 - \left[1-\left(2ln(x+2)\right)_0^1\right]$

$= ln\,3 - \left[1-\left(2\,ln\,3 - 2\,ln\,2\right)\right]$

$= ln\,3 - 1 + 2\,ln\,3 - 2\,ln\,2$

$= 3\,ln\,3 - 2\,ln\,2 - 1$

$= ln\left(\frac{27}{4}\right) - ln\,e = ln\left(\frac{27}{4e}\right)$

$\therefore \quad \lim_{n\to\infty} \frac{P_n}{n} = \left(\frac{27}{4e}\right)$ **Ans.** ]

**21** Let f be an injective function such that $f(x)\,f(y) + 2 = f(x) + f(y) + f(xy)$ for all non negative real x & y with $f'(0) = 0$ & $f'(1) = 2 \neq f(0)$. Find f(x) & show that, $3\int f(x)\,dx - x\,(f(x) + 2)$ is a constant.

Sol $g(x)3\int f(x).dx - x(f(x)+2)$

$\Rightarrow g'(x) = 3f(x) - f(x) - xf'(x) - 2$

$\Rightarrow g'(x) = 2f(x) - xf'(x) - 2 \qquad ...(1)$

$f(x).f(y) + 2 = f(x) + f(y) + f(xy)$

Partilly differentiating w.r.t.x,

$f'(x)\,f(y) = f'(x) + yf'(xy)$

Putting $x = 1$, $f'(1)\cdot f(y) = f'(1) + yf'(y)$

$(\because f'(1) = 2)$

Hence, $2f(y) - yf'(y) - 2 = 0 \quad (\forall\, y \in R)$ ...(2)

Using (1) & (2), $g'(x) = 0 \quad (\forall\, x \in R)$

Hence, $g(x)$ is a constant function.

**22** Prove that $\sin x + \sin 3x + \sin 5x + .... + \sin(2k-1)\,x = \frac{\sin^2 kx}{\sin x}$, $k \in N$ and hence prove that, $\int_0^{\pi/2} \frac{\sin^2 kx}{\sin x}\,dx = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ...... + \frac{1}{2k-1}$.

Sol We know,

$e^{ix} = \cos x + i \sin x$

$$e^{i3x} = \cos 3x + i\sin 3x$$

$$e^{i(2k-1)x} = \cos(2k-1)x + i\sin(2k-1)x$$

Adding all,

$$\underbrace{e^{ix} + e^{i3x} + .... + e^{i(2r-1)x}}_{\left(\text{G.P with common ratio } e^{i2x}\right)} = \sum_{r=1}^{k}\cos(2r-1)x$$

$$\underbrace{+i\sum_{r=1}^{k}\sin(2r-1)x}_{\substack{\uparrow \\ \text{S,say}}}$$

Hence,

$$S = \frac{\left(e^{ix}\right)\left(\left(e^{i2x}\right)^k - 1\right)}{\left(e^{i2x}-1\right)} \quad = \frac{\left(e^{2kx}-1\right)}{\left(e^{ix}-e^{-ix}\right)}$$

$$= \frac{\left(e^{i(2kx)}-1\right)}{\left(2i\sin x\right)} \quad = (i)\left(\frac{1-e^{i(2kx)}}{2\sin x}\right)$$

$$\sum_{r=1}^{k}\sin(2r-1)x = \left(\frac{1-\cos 2kx}{2\sin x}\right)$$

Hence,

$$\frac{2\sin^2 kx}{2\sin x} = \left(\frac{\sin^2 kx}{\sin x}\right) = \sin x + \sin 3x + ..... + \sin(2k-1)x = \left(\frac{\sin^2 kx}{\sin x}\right)$$

Now, $\int_0^{\frac{\pi}{2}} \frac{\sin^2 kx}{\sin x}\cdot dx = \int_0^{\frac{\pi}{2}}\left(\sin x + \sin 3x + .... + \sin(2k-1)x\right).dx$

$$= \left(\cos x + \frac{\cos 3x}{3} + .... + \frac{\cos(2k-1)x}{(2k-1)}\right)_{\frac{\pi}{2}}^{0} = \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + .... + \frac{1}{2k-1}\right).$$

**24**

Sol $(1-x)^n = \sum_{k=0}^{n}(-1)^k\ {}^nC_k\ x^k$

$$\Leftrightarrow x^m(1-x)^n = \sum_{k=0}^{n}(-1)^k\ {}^nC_k\ x^{m+k}$$

Integrating the equation from 0 to 1, $\int_0^1 x^m(1-x)^n.dx = \sum_{k=0}^{n}\frac{(-1)^k\ {}^nC_k}{m+k+1}$ .....(1)

$|||^{\text{ly}}$, consideing $(1-x)^m$, $\int_0^1 x^n(1-x)^m_{dx} = \sum_{k=0}^{m}\frac{(-1)^k\ {}^mC_k}{n+k+1}$ .....(2)

But, $\int_0^1 x^n(1-x)^m.dx = \int_0^1 x^m(1-x)^n.dx$ ....(3)

$\left(\int_a^b f(x) = \int_a^b f(a+b-x).dx\right)$

(1),(2),(3)

$\Rightarrow \sum_{k=0}^{n} \frac{(-1)^k\ {}^nC_k}{m+k+1} = \sum_{k=0}^{m} \frac{(-1)^k\ {}^mC_k}{n+k+1}.$

**25**

[Sol.

**(a)** in (0, 1) $4 - x^2 - x^3 < 4 - x^2$

$\frac{1}{4-x^2-x^3} > \frac{1}{4-x^2}$

$\therefore \quad \frac{1}{\sqrt{4-x^2-x^3}} > \frac{1}{\sqrt{4-x^2}}$

$\therefore \quad \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} > \int_0^1 \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\frac{x}{2}\Big]_0^1 = \frac{\pi}{6} \Rightarrow \frac{\pi}{6} < I$

Again $4 - x^2 - x^3 > 4 - 2x^2$ in (0, 1)

$\frac{1}{\sqrt{4-x^2-x^3}} < \frac{1}{\sqrt{4-2x^2}}$

$I < \frac{1}{\sqrt{2}}\int_0^1 \frac{1}{\sqrt{2-x^2}} = \frac{1}{\sqrt{2}}\sin^{-1}\frac{x}{\sqrt{2}}\Big]_0^1 = \frac{\pi}{4\sqrt{2}} = \frac{\pi\sqrt{2}}{8} \Rightarrow \frac{\pi}{6} < I < \frac{\pi\sqrt{2}}{8}$

**(b)** $I = \int_0^2 e^{x^2-x}.dx$

Let, $f(x) = e^{x^2-x}$

$f'(x) = \left(e^{x^2-x}\right)(2x-1)$

in $x \in (0,2)$

$f_{\min} = f\left(\frac{1}{2}\right) = e^{-\frac{1}{4}}$

$f_{\max} = e^2 \max\{f(0), f(2)\}$

Hence, $\int_0^2 f_{\min} < I < \int_0^2 f_{\max}$

$\boxed{2e^{-\frac{1}{4}} < I < 2e^2}.$

**(c)** $I=\int_0^{2\pi}\frac{dx}{10+3\cos x}=2\int_0^{\pi}\frac{dx}{10+3\cos x}$

( cos x repeats it self i.e. takes same value again in $(\pi,2\pi)$)

Let, $\frac{x}{2}=t$

$$I=4\int_0^{\frac{\pi}{2}}\frac{dt}{10+3\cos 2t}$$

$$=4\int_0^{\frac{\pi}{2}}\frac{dt}{10+3\left(\frac{1-\alpha^2}{1+\alpha^2}\right)}\quad(\alpha=\tan t)$$

$$=4\int_0^{\frac{\pi}{2}}\frac{(1+\alpha^2).dt}{(13+7\alpha^2)}$$

Let , $m=\tan t$

$dm=\sec^2 t=(1+\alpha^2)dt$

$$I=\frac{4}{7}\int_0^{\infty}\frac{dm}{\frac{13}{7}+m^2}=\left(\frac{4}{7}\right)\left(\sqrt{\frac{7}{13}}\right)\left(\tan^{-1}\left(\sqrt{\frac{7}{13}}m\right)\right)_0^{\infty}=\left(\frac{4}{\sqrt{91}}\right)\left(\frac{\pi}{2}\right)$$

$$\boxed{I=\left(\frac{2\pi}{\sqrt{91}}\right)}$$

$\sqrt{49}<\sqrt{91}<\sqrt{169}$ ( $\sqrt{x}$ is function on $x>0$)

Hence, $\frac{2\pi}{13}<I<\frac{2\pi}{7}$.

**(d)** $I=\int_0^{2}\frac{dx}{2+x^2}=\frac{1}{\sqrt{2}}\left(\tan^{-1}\frac{x}{\sqrt{2}}\right)_0^2 \quad =\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\right)$

Let , $f(x)=\frac{1}{1+x^2}$

$f'(x)=\frac{-2x}{(2+x^2)^2}<0\quad \forall x>0$ or $x\in(0,2)$,

$f_{\min}=f(2)=\left(\frac{1}{6}\right)$ , $f_{\max}=f(0)=\left(\frac{1}{2}\right)$

$$\int_0^2 f_{\min} < I < \int_0^2 f_{\max}$$
$$\underset{\frac{1}{3}}{\uparrow} \qquad \underset{1}{\uparrow} \quad .$$

**26**

Sol $$f(x) = x + \int_0^1 xy^2 . f(y).dy + \int_0^1 x^2 y\, f(y).dy$$

$$f(x) = x + x\left(\int_0^1 y^2 . f(y).dy\right)$$

$$+x^2\left(\int_0^1 y.f(y).dy\right) \qquad \text{.. ..(1)}$$

Hence, $f(x) = Ax^2 + Bx$ (A,B are real constant)

Using (1),

$$Ax^2 + Bx = x + x\int_0^1 y^2\left(Ay^2 + By\right).dy$$

$$+x^2\int_0^1 y\left(Ay^2 + By\right).dy$$

$$= x + x\left[\frac{A}{5} + \frac{B}{4}\right] + x^2\left[\frac{A}{4} + \frac{B}{3}\right]$$

$$\Leftrightarrow x^2\left[\frac{3A}{4} - \frac{B}{3}\right] + x\left[\frac{3B}{4} - \frac{A}{5} - 1\right] = 0 \qquad \text{....(2)}$$

(2) is true $\forall x \in R$

Hence, $\frac{3A}{4} - \frac{B}{3} = 0$ ...(i) and,

$\frac{3B}{4} - \frac{A}{5} - 1 = 0$ ...(ii)

Solving (i) , (ii) , we get,

$$A = \left(\frac{81}{119}\right)$$

$$B = \left(\frac{180}{119}\right)$$

Hence, $\boxed{f(x) = \frac{80x^2}{119} + \frac{180x}{119}}$.

**Q.27**

**Sol.** $I_1 = \int_{-1}^{1} \{x+1\}\{x^2+2\} + \{x^2+2\}\{x^3+4\}dx$

$= \int_{-1}^{1} \{x\}\{x^2\} + \{x^2\}\{x^3\}dx$ $\qquad \because \{x+I\} = \{x\}$

$= \int_{-1}^{1} (x-[x])(x^2-[x^2]) + (x^2-[x^2])(x^3-[x^3])dx$

$\int_{-1}^{0} (x+1)(x^2) + (x^2)(x^3+1)\,dx + \int_{0}^{1} (x.x^2 + x^2.x^3)\,dx$

$= \int_{-1}^{0} x^2(x^3+x+2)\,dx + \int_{0}^{1} (x^3+x^5)\,dx$

$= \int_{-1}^{0} (x^5+x^3+2x^2)\,dx + \int_{0}^{1} (x^3+x^5)dx = \left(\frac{x^6}{6}+\frac{x^4}{4}+\frac{2x^3}{3}\right)_{-1}^{0} + \left(\frac{x^4}{4}+\frac{x^5}{5}\right)_{0}^{1}$

$= \frac{2}{3}$

**Q.28**

**Sol.** $I = \int_{1}^{16} \tan^{-1}\left(\sqrt{\sqrt{x}-1}\right) dx$

$\sqrt{x} = \sec^2\theta$ $\qquad$ when $x = 1$ $\sec^2\theta = 1$

$\sec\theta = 1$

$x = \sec^4\theta$ $\qquad$ $\theta = 0$

$dx = 4\sec^2\theta.\sec\theta\tan\theta\,d\theta$ $\qquad$ $x = 16$ $\quad \sec^2\theta = \sqrt{16}$

$\sec^2\theta = 4$

$\sec\theta = 2 \Rightarrow \theta = \pi/3$

$I = \int_{0}^{\frac{\pi}{3}} \tan^{-1}\sqrt{\sec^2\theta - 1}\ 4\sec^4\theta\tan\theta\,d\theta$

$$= 4\int_0^{\frac{\pi}{3}} \theta \sec^4\theta \tan\theta \, d\theta$$

$$= 4\int_0^{\frac{\pi}{3}} \theta \sec^3\theta(\sec\theta \tan\theta)d\theta$$

$$= 4\left(\left[\theta\frac{\sec^4\theta}{4}\right]_0^{\pi/3} - \int_0^{\frac{\pi}{3}}\frac{\sec^4\theta}{4}d\theta\right) \text{ (using by parts)}$$

$$= \left(\frac{\pi}{3}(2)^4\right) - \int_0^{\frac{\pi}{3}} \sec^4\theta \, d\theta$$

$$= \frac{16\pi}{3} - \int_0^{\frac{\pi}{3}} \sec^4\theta \, d\theta$$

$$= \frac{16\pi}{3} - \int_0^{\frac{\pi}{3}} \sec^4 d\theta$$

$$= \frac{16\pi}{3} - \int_0^{\frac{\pi}{3}} \sec^2\theta.(1+\tan^2\theta)d\theta$$

$$= \frac{16\pi}{3} - \left(\int_0^{\frac{\pi}{3}} \sec^2\theta + \int_0^{\frac{\pi}{3}} \sec^2\theta \tan^2\theta d\theta\right)$$

$$= \frac{16\pi}{3} - \left[\tan\theta + \frac{\tan^3\theta}{3}\right]_0^{\frac{\pi}{3}}$$

$$= \frac{16\pi}{3} - \left(\sqrt{3} + \frac{3\sqrt{3}}{3}\right)$$

$= \frac{16\pi}{3} - 2\sqrt{3}$

**Q.29**

**Sol.** put 2x = t

dx = dt/2

$$= \int_0^{2\pi} \frac{dx}{2 + \frac{2\tan x}{1+\tan^2 x}}\, dx \;=\; \frac{1}{2}\int_0^{2\pi} \frac{\sec^2 x}{\tan^2 x + \tan x + 1} dx$$

**Q.30**

**Sol.** $I'(a) = \int_0^a \frac{x}{(1+ax)(1+x^2)} dx$

**Q.31**

**Sol.** $$= \int_0^{\frac{\ell n3}{2}} \frac{e^x + 1}{e^{2x}+1}\, dx$$

$$= \int_0^{\frac{\ell n3}{2}} \frac{e^x}{e^{2x}+1}\, dx \;+\; \int_0^{\frac{\ell n3}{2}} \frac{1}{e^{2x}+1}\, dx$$

$$= \int_0^{\frac{\ell n3}{2}} \frac{e^x}{e^{2x}+1} dx + \int_0^{\frac{\ell n3}{2}} \frac{1}{e^{2x}+1}\, dx$$

put $e^x = t$

$e^x\, dx = dt$

$$= \int_1^{\sqrt{3}} \frac{dt}{t^2+1} + \int_0^{\frac{\ell n3}{2}} \frac{e^{-2x}}{1+e^{-2X}}\, dx$$

put $1 + e^{-2x} = p - 2e^{-2x}\, dx = dp$

$$= \tan^{-1} t \int_1^{\sqrt{3}} \frac{1}{2}\int_2^{4/3} \frac{dt}{t}$$

$$= \frac{\pi}{3} - \frac{\pi}{4} - \frac{1}{2}\ell nt\Big|_2^{4/3}$$

$= \frac{\pi}{12} - \frac{1}{2}\left(\ell n \frac{4}{3}\right)$

$= \frac{1}{2}\left[\frac{\pi}{6} - \ell n \frac{2}{3}\right]$ **Ans**

**Q.32**

**Sol.** $I = \int_0^{2\pi} \frac{x^2 \sin x}{8 + \sin^2 x} dx$

$= \int_0^{2\pi} \frac{(2\pi - x)^2 \sin(2\pi - x)}{8 + \sin^2(2\sin x)} dx$

$I = \int_0^{2\pi} \frac{(-x^2 + 4\pi x - 4\pi^2)\sin x}{8 + \sin^2 x} dx$ ...(2)

$2I = \int_0^{2\pi} \frac{(4\pi x - 4\pi^2)\sin x}{8 + \sin^2 x} dx$

$= 4\pi \int_0^{2\pi} \frac{x \sin x}{8 + \sin^2 x} dx - 4\pi^2 \int_0^{2\pi} \frac{\sin x}{8 + \sin^2 x} dx$

$I = 2\pi \int_0^{2\pi} \frac{x \sin x}{8 + \sin^2 x} dx - 2\pi^2 \int_0^{2\pi} \frac{\sin x}{8 + 1 - \cos^2 x} dx$

Let $I_1 = \int_0^{2\pi} \frac{x \sin x}{8 + \sin^2 x} dx$

let x = n + t dx = dt

$I_1 = \int_{-\pi}^{\pi} \frac{(n + t)(-\sin t)}{8 + \sin^2 t} dt$

$I_1 = -\pi \int_{-n}^{\pi} \frac{\sin t}{8 + \sin^2 t} dt - \int_{-n}^{\pi} \frac{t \sin t}{8 + \sin^2 t} dt$

$I_1 = -2\int_0^{\pi} \frac{t \sin t}{8 + \sin^2 t} dt$

$$I_1 = -2\int_0^{\pi} \frac{(\pi + t)\sin t}{8+\sin^2 t}dt$$

$$2I_1 = -\ 2\int_0^{\pi} \frac{\pi \sin t}{8+\sin^2 t}dt$$

$$2I_1 = -2\pi \int_0^{\pi} \frac{\sin t}{8+\sin^2 t}dt$$

$$I_1 = -\pi \int_0^{\pi} \frac{\sin t}{8+\sin^2 t}dt$$

$$= -\pi.2\int_0^{\frac{\pi}{2}} \frac{\sin t}{9-\cos^2 t}dt$$

$$= 2\pi\ \frac{1}{6}\left[\log\left(\frac{3+\cos t}{3-\cos t}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{3}\left(\ell \pi 1 - \log\left(\frac{4}{2}\right)\right)$$

$$= \left(-\frac{\pi}{3}\log 2\right)$$

$$I_2 = \int_0^{2\pi} \frac{\sin x}{9-\cos^2 x}dx$$

$$I_2 \int_0^{2\pi} \frac{\sin(2\pi - x)}{9-\cos^2(2\pi - x)}dx$$

$$I_2 = -\int_0^{2\pi} \frac{\sin x}{9-\cos^2 x}dx$$

$I_2 = 0$

so ultimate

$I = 2\pi I_1 - 2\pi^2 I_2$

$$= 2\pi\left(-\frac{\pi}{3}\log 2\right)$$

$$= -\frac{2\pi^2}{3}\log 2$$ **Ans.**

**Q.33**

**Sol.** $I = \frac{1}{2}\int_0^1 (2\sin\alpha x.\sin\beta x)\,dx = \frac{1}{2}\int_0^1 (\cos(\alpha-\beta)x - \cos(\alpha+\beta)x)\,dx$

$$= \frac{1}{2}\left[\frac{\sin(\alpha-\beta)}{\alpha-\beta}x - \frac{\sin(\alpha+\beta)x}{\alpha+\beta}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{\sin(\alpha-\beta)}{\alpha-\beta} - \frac{\sin(\alpha+\beta)}{\alpha+\beta}\right]$$

Now

$$\left.\begin{array}{l} 2\alpha = \tan\alpha \\ 2\beta = \tan\beta \end{array}\right\} \Rightarrow \begin{array}{l} 2(\alpha-\beta) = \tan x - \tan\beta \\ 2(\alpha+\beta) = \tan x + \tan\beta \end{array}$$

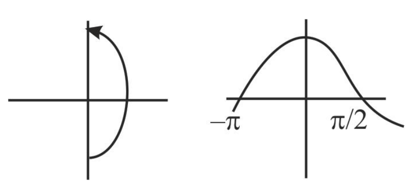
$$\therefore\ 2(\alpha-\beta) = \frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}\ \&\ 2(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$$

put these values

$I = \cos\alpha\cos\beta - \cos\alpha\cos\beta = 0$ **Ans**

**Q.34**

**Sol.**




$$= \int_0^p \cos x\,dx + \int_p^{p+q\pi} |\cos x|\,dx$$

$$= \sin x \int_0^p + q\int_0^{\pi} |\cos x|\, dx$$

$$= \sin p + q\left[\int_0^{\pi/2} \cos x\, dx - \int_{\pi/2}^{\pi} \cos x\, dx\right]$$

$= \sin p + q\ (1 + 1)$

$= 2q + \sin p$ **Ans**

**Q.35.**

**Sol.** $$f(\theta) \int_0^1 \frac{\tan^{-1} dx}{x^2 + 2x\cos\theta + 1}$$

$x = \tan\phi$ $dx = \sec^2\phi\, d\phi$

$$= \int_0^{\frac{\pi}{2}} \frac{\phi \sec^2\phi\, d\phi}{\tan^2\phi + 2\tan\phi\cos\theta + 1}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\phi\sec^2\phi}{\sec^2\phi + 2\tan\phi\cos\theta} d\phi$$

$$= \int_0^{\frac{\pi}{2}} \frac{\phi.\frac{1}{\cos^2\phi}}{\frac{1}{\cos^2\phi} + \frac{2\sin\phi\cos\theta}{\cos\phi}} d\phi$$

$$= \int_0^{\frac{\pi}{2}} \frac{\phi}{1 + 2\sin\phi\cos\phi\cos\theta} d\phi$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\phi}{1 + (\sin 2\phi)\cos\theta} d\phi$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2} - \phi}{1 + (\sin 2\phi)\cos\theta} d\phi$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}}{1+(\sin 2\phi)\cos\theta} d\phi$$

$$I = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \frac{1}{1+(\sin 2\phi)\cos\theta} d\phi$$

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{2\tan\phi}{1+\tan^2\phi}\cos\theta} d\theta$$

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \frac{\sec^2\phi}{\tan^2\phi + 2\tan\phi\cos\theta + 1} d\phi$$

let $\tan\phi = y$

$\sec^2\phi \; d\theta = dy$

$$= \frac{\pi}{4}\int_0^{\infty} \frac{dy}{y^2 + 2y\cos\theta + 1}$$

$$= \frac{\pi}{4} + \frac{\theta}{\sin\theta} = \frac{\pi\theta}{4\sin\theta}$$

**Q.36**

**Sol.** $$I = \int_0^{\pi} \frac{x\sin^3 x}{4-\cos^2 x} dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin^3(\pi - x)dx}{4-\cos^2(\pi - x)}$$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin^3 x}{4-\cos^2(x)} dx$$

$$2I = \int_0^{\pi} \frac{\pi\sin^3 x}{4-\cos^2 x} dx$$

$$I = \frac{\pi}{2}.2\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{4-\cos^2 x}dx$$

$$= \pi\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{4-\cos^2 x}dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \frac{(1-\cos^2 x)\sin x}{(4-\cos^2 x)}dx$$

Let cosx = t

– sin x dx = dt

$$= -\pi\int_1^0 \frac{(1-t^2)}{(4-t^2)}dt$$

$$= \pi\int_0^1 \frac{(4-t^2)-3}{(4-t^2)}dt$$

$$= \pi\left[t - \frac{3}{2-t}\log\left(\frac{2+t}{2-t}\right)\right]_0^1$$

$$\pi\left(1-\frac{3}{4}\log 3\right)$$

$$= \pi\left(1-\frac{2\log b}{c}\right)$$

a = 3 b = 3

c = 4

= 3 + 3 + 4 = 10

**Q.37**

**Sol.** $I = \int_0^{\pi/2} \tan^{-1}\left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right]dx$

$$\int_0^{\frac{\pi}{2}} \tan^{-1} \frac{\left|\sin\frac{x}{2}+\cos\frac{x}{2}\right| + \left|\sin\frac{x}{2}-\cos\frac{x}{2}\right|}{\left|\sin\frac{x}{2}+\cos\frac{x}{2}\right| - \left|\sin\frac{x}{2}-\cos\frac{x}{2}\right|}$$

$$= \int_0^{\frac{\pi}{2}} \tan^{-1}\left(\frac{\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}+\cos\frac{x}{2}}{\sin\frac{x}{2}+\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \tan^{-1}\left(\cot\frac{x}{2}\right)dx$$

$$I = \int_0^{\frac{\pi}{2}} \tan^{-1}\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)dx$$

$$0 < \frac{x}{2} < \frac{\pi}{4}$$

$$0 > -\frac{x}{2} > -\frac{\pi}{4}$$

$$\frac{\pi}{2} > \frac{\pi}{2} - \frac{x}{2} > \frac{\pi}{4}$$

$$\text{so } I = \int_0^{\frac{\pi}{2}} \frac{\pi}{2} - \frac{x}{2}\,dx$$

$$= \frac{\pi}{2}\times\frac{\pi}{2} - \frac{1}{4}\times\frac{\pi^2}{4}$$

$$= \frac{\pi^2}{4} - \frac{\pi^2}{16} = \frac{3\pi^2}{16}$$ **Ans.**

**Q.38**

**Sol.** $$\int_{\frac{\sqrt{3a^2+b^2}}{2}}^{\sqrt{\frac{a^2+b^2}{2}}} \frac{x\,dx}{\sqrt{(x^2-a^2)\,(b^2-x^2)}}\,dx$$

Let $x^2 = a^2\cos^2\theta + b^2\sin^2\theta$ $\qquad$ $2x\,dx = (-a^2\,2\cos\theta\,\sin\theta + b^2 2\sin\theta\cos\theta)\,d\theta$

$x^2 - a^2 = a^2\cos^2\theta + b^2\sin^2\theta - a^2$

$= b^2\sin^2\theta - a^2\sin^2\theta$

$= (b^2 - a^2)\sin^2\theta$

$= b^2 - b^2\cos^2\theta - a^2\sin^2\theta$

$= b^2\cos^2\theta - a^2\cos^2\theta$

$= (b^2 - a^2)\cos^2\theta$

$x\,dx = (b^2 - a^2)\sin 2\theta\, d\theta$

who $x^2 = \dfrac{3a^2 + b^2}{4}$

$0^2\cos^2\theta + b^2\sin^2\theta = \dfrac{3a^2 + b^2}{4}$

$4a^2\cos^2\theta + 4b^2\sin^2\theta = 3a^2 + b^2$

$4a^2 - 4a^2\sin^2\theta + 4b^2\sin^2\theta = 3a^2 + b^2$

$(a^2 - b^2) = 4(a^2 - b^2)\sin^2\theta$

$\sin^2\theta = 1/4$

$\sin\theta = \dfrac{1}{2}$

$\theta = \dfrac{\pi}{6}$

$x^2 = \dfrac{a^2 + b^2}{2}$

$a^2\cos^2\theta + b^2\sin^2\theta = \dfrac{a^2 + b^2}{2}$

$2a^2\cos^2\theta + 2b^2\sin^2\theta = a^2 + b^2$

$2a^2 - 2a^2\sin^2\theta + 2b^2\sin^2\theta = a^2 + b^2$

$(a^2 - b^2) = 2(a^2 - b^2)\sin^2\theta$

$\sin^2\theta = \dfrac{1}{2}$

$\sin\theta = \dfrac{1}{\sqrt{2}}$

$\theta = \dfrac{\pi}{4}$

$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{(b^2 - a^2)\sin 2\theta d\theta}{\sqrt{(b^2 - a^2)\sin^2\theta (b^2 - a^2)}\cos^2\theta}$$

$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{(b^2 - a^2)\sin 2\theta\, d\theta}{(b^2 - a^2)\sin\theta\cos\theta}$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} d\theta = \left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \left(\frac{3\pi - 2\pi}{12}\right) = \frac{\pi}{12}$$ **Ans.**

**Q.39**

**Sol.** $x^2 + 2x = k + \int_0^1 |t + k|\, dt$

**CaseI**, $k \ge 0$ $|t + k| = \begin{cases} (t+k), (t \ge -k) \\ -(t+k), \ t \le -k \end{cases}$

$x^2 + 2x = k + \int_0^1 (t+k)\,dt$

$= k + \left[\frac{t^2}{2} + kt\right]_0^1$

$= k + \frac{1}{2} t + k = 2k + \frac{1}{2}$

$2x^2 + 4x = 4k + 1$

$2x^2 + 4x - (4k + 1) = 0$

$D = (4)^2 + 4.2\,(4k + 1)$

$= 16 + 8(4k + 1)$

$= 8\,(2 + 4k + 1)$

$= 8\,(4k + 3)$

as $k \ge 0$ $D > 0$

$\Rightarrow$ Roots are real & enequal.

**Case II** $k < 0$, let $k = -a$, $a > 0$

$x^2 + 2x = -a + \int_0^1 |t - a|\,dt$

Now $|t - a| =$ $(t - a), t > a$

$-(t - a), t < a$

Now code $0 < a < 1$

$x^2 + 2x = -a + \int_0^a -(t-a)dt + \int_a^1 (t-a)dt$

$= -a + \left(-\frac{t^2}{2} + at\right)_0^a + \left[\frac{t^2}{2} - at\right]_a^1$

$x^2 + 2x = -a + \left(-\frac{a^2}{2} + a^2 + \frac{1}{2} - a - \frac{a^2}{2} + a^2\right)$

$= -a + \left(2a^2 - a^2 - a + \frac{1}{2}\right)$

$= - a + a^2 - a + \frac{1}{2}$

$x^2 + 2x = a^2 - 2a + \frac{1}{2}$

$2x^2 + 4x = 2a^2 - 4a + 1$

$2x^2 + 4x - (2a^2 - 4a + 1) = 0$

$D = 16 + 8(2a^2 - 4a + 1)$

$= 8(2 + 2a^2 - 4a + 1)$

$= 8 (2a^2 - 4a + B)\ 16 - 4.2.3 < 0$

$\Rightarrow D > 0 \;\; D \in (0,1)$

roots are real & unequal

$a \geq 1$

$x^2 + 2x = - a \int_0^1 (t - a)\,dt$

$= - a - \left(\frac{t^2}{2} - at\right)_0^1$

$= - a - \left(\frac{1}{2} - a\right)$

$= - a + \frac{1}{2} + a$

$= - \frac{1}{2}$

$2x^2 + 4x + 1 = 0$

$D > 16 - 8 \geq 0$

Roots one real & unequal

as real & unequal $\forall\, k \in B$

**Q.40**

**Sol.** $\int_{-1}^{1} \frac{2x^{332} + x^{998}}{1 + x^{666}}\,dx + \int_{-1}^{1} \frac{4x^{1668} \sin x^{691}}{1 + x^{666}}\,dx$

$= 2\int_0^1 \frac{x^{332}(1 + 1 + x^{666})}{1 + x^{666}}\,dx$

$$= 2\int_0^1 \frac{x^{332}}{1+x^{666}}\,dx + 2\int_0^1 x^{332}\,dx$$

$$= 2\int_0^1 \frac{x^{332}}{1+(x^{333})^2}\,dx + 2\int_0^1 x^{332}dx$$

**Q.41**

**Sol.** $$I = \int_0^{\pi} \frac{x^2 \sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$$

Let $x = \frac{\pi}{2}+t$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}+t\right)^2 \sin\left(\frac{2\pi}{2}+2t\right)\sin\left(\frac{\pi}{2}\cos\left(\frac{\pi}{2}+t\right)\right)}{2\left(\frac{\pi}{2}+t\right)-\pi}dt$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}+t\right)^2 \sin 2t \sin\left(\frac{\pi}{2}\sin t\right)}{t}dt$$

$$= \frac{1}{2}\left[\frac{\pi^2}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin 2x \sin\left(\frac{\pi}{2}+t\right)}{t}dt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{t^2 + 2t\sin\left(\frac{\pi}{2}\sin t\right)}{t}dt + \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin 2t + \sin\left(\frac{\pi}{2}\sin t\right)}{t}dt\right.$$

$$= \frac{\pi}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin 2t \sin\left(\frac{\pi}{2}\sin t\right)dt$$

$$I = \frac{\pi}{2} - 2\int_0^{\frac{\pi}{2}} \sin 2t \sin\left(\frac{\pi}{2}\sin t\right)dt$$

$$\pi\int_0^{\frac{\pi}{2}} \sin 2t \sin\left(\frac{\pi}{2}\sin t\right)dt$$

Let $\sin t = y$

$\cos t\, dt = dy$

$$I = \pi\int_0^1 2y\sin\left(\frac{\pi}{2}y\right)dy$$

$$= 2\pi\int_0^1 y\sin\left(\frac{\pi}{2}y\right)dy$$

$$= 2\pi\left(\left[-\frac{y-\cos\frac{\pi}{2}y}{\frac{\pi}{2}}\right]_0^1 + \int_0^1 \frac{\cos\frac{\pi}{2}y}{\frac{\pi}{2}}dy\right)$$

$$= 2\pi\left[-\frac{y\cos\frac{\pi}{2}y}{\frac{\pi}{2}} + \frac{\sin\frac{\pi}{2}y}{\left(\frac{\pi}{2}\right)^2}\right]_0^1$$

$$= 2\pi\left(\frac{1}{\left(\frac{\pi}{2}\right)^2} - (0)\right)$$

$$- \ 2\pi \times \frac{1}{\pi^2} \times 4 = \frac{8}{\pi}$$

**Q.42**

**Sol.** $$\int_0^\infty \frac{dx}{x^2+2x\cos\theta+1}$$

$$\int_0^\infty \frac{dx}{x^2+2x\cos\theta+\cos^2\theta+1-\cos^2\theta}$$

$$\int_0^\infty \frac{dx}{(x+\cos\theta)^2+(\sin\theta)^2}$$

$$\frac{1}{\sin\theta}\left[\tan^{-1}\frac{x+\cos\theta}{\sin\theta}\right]_0^\infty$$

$$\frac{1}{\sin\theta}\left(\frac{\pi}{2}-\tan^{-1}\frac{\cos\theta}{\sin\theta}\right)=\frac{1}{\sin\theta}\left(\frac{\pi}{2}-\left(\frac{\pi}{2}-\theta\right)\right)=\frac{\theta}{\sin\theta}$$

RHS $2\int_0^1 \frac{dx}{x^2+2x\cos\theta+1}=\frac{2}{\sin\theta}.\left[\tan^{-1}\frac{\cos\theta}{\sin\theta}\right]_0^1$

$$=\frac{2}{\sin\theta}\left(\tan^{-1}\frac{1+\cos\theta}{\sin\theta}-\tan^{-1}\frac{\cos\theta}{\sin\theta}\right)$$

$$=\frac{2}{\sin\theta}\left(\tan^{-1}\cos\frac{\theta}{2}-\tan^{-1}\cos\theta\right)$$

$$=\frac{2}{\sin\theta}\left(\frac{\pi}{2}-\frac{\theta}{2}-\frac{\pi}{2}+\theta\right)=\frac{2}{\sin\theta}\left(\frac{\theta}{2}\right)=\frac{\theta}{\sin\theta}$$

LHS = RHS

**Method II**

$$\int_0^\infty \frac{dx}{x^2+2x\cos\theta+1}=\int_0^1 \frac{dx}{x^2+2x\cos\theta+1}+\int_1^9 \frac{dx}{x^2+2x\cos\theta+1}$$

Now $\int_0^\infty \frac{dx}{x^2+2x\cos\theta+1}$

$x=\frac{1}{t}$

$dx=-\frac{1}{t^2}dt$

$$\int_1^0 \frac{\frac{-1}{t^2}dt}{\frac{1}{t^2}+\frac{2\cos\theta}{t}+1}$$

$$\int_1^0 \frac{-1}{1+2t\cos\theta+t^2}=\int_0^1 \frac{dt}{x^2+2t\cos\theta+1}$$

$$=\int_0^1 \frac{dx}{x^2+2x\cos\theta+1}$$

$$\text{so} \int_0^{\infty} \frac{dx}{x^2 + 2x\cos\theta + 1} = 2\int_0^{1} \frac{dx}{x^2 + 2x\cos\theta + 1}$$

**Q.43**

**Sol.** $I = \lim\limits_{n\to\infty} \frac{1}{n^2} \sum\limits_{k=0}^{n-1} \left[ k \int_k^{k+1} \sqrt{(x-k)\ (k+1-x)}\, dx \right]$

Now $I_1 = \int_k^{k+1} \sqrt{(x-k)((k+1)-x)}\, dx$

$x = k\cos^2\theta + (k + 1)\sin^2\theta$

$dx = -2k\cos\theta\sin\theta + (kx)\ 2\sin\theta\cos\theta\, d\theta$

$= 2(k + 1 - k)\sin\theta\cos\theta\, d\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

$x - k = k\cos^2\theta + (k + 1)\sin^2\theta - k$

$= (k + 1)\sin^2\theta - k(1 - \cos^2\theta)$

$= (k + 1 - k)\sin^2\theta$

$= \sin^2\theta$

$(k + 1) - x = (k + 1) - k\cos^2\theta - (k + 1)\sin^2\theta$

$= 1 + k\sin^2\theta - k\sin^2\theta - \sin^2\theta$

$= \cos^2\theta$

where $x = k$ $\quad$ $k = k\cos^2\theta + (k + 1)\sin^2\theta$

$k\sin^2\theta = (k+1)\sin^2\theta$

$k\sin^2\theta = k\sin^2\theta + \sin^2\theta$

$\sin^2\theta = 0$

$\theta = 0$

where

$x = k + 1$ $\quad$ $k + 1 = k\cos^2\theta + (k + 1)\sin^2\theta$

$(k + 1)\cos^2\theta = k\cos^2\theta$

$\cos^2\theta = 0$

$\cos\theta = 0$ $\qquad$ $\theta > \frac{\pi}{2}$

$\text{so } I_1 = \int_0^{\frac{\pi}{2}} (\sqrt{\sin^2\theta\ \cos^2\theta})\ 2\sin\theta\cos\theta\, d\theta$

$$= \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta.2\sin\theta\cos\theta d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (\sin 2\theta)^2 d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{2} d\theta$$

$$= \frac{1}{4}\left(\theta - \frac{\sin 4\theta}{2}\right)_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left(\frac{\pi}{2}\right)$$

so $I = \underset{x\to\infty}{\text{Lim}} \frac{1}{n^2} \sum_{k=0}^{n-1} \left[ k\int_k^{k+1} \sqrt{(x-k)\ (k+1)-x)dx} \right]$

$$= \underset{x\to\infty}{\text{Lim}} \frac{1}{n^2} \sum_{k=0}^{n-1} \left( k.\frac{n}{8} \right)$$

$$= \underset{x\to\infty}{\text{Lim}} \frac{1}{n} \sum_{k=0}^{n-1} \left( \frac{k}{n} \right)$$

$$= \frac{n}{8}\int_0^1 x dx = \frac{n}{8} \times \frac{1}{2} = \frac{n}{16}$$

**Q.44**

**Sol.** $\int_0^{\infty} f\left(\frac{a}{x} + \frac{x}{a}\right).\frac{\ell nx}{x} dx$

$x = a\tan\theta$

$I = \int_0^{\frac{n}{2}} f(\tan\theta + \cot\theta)\ \frac{\log(a\tan\theta)}{a\tan\theta}\ a\sec^2\theta\, d\theta$

$$= \int_0^{\frac{n}{2}} f(\tan\theta + \cot\theta)\ \log\ \frac{\tan(\theta)}{\frac{\sin\theta}{\cos\theta}} \frac{1}{\cos^2\theta} d\theta$$

$$I = 2\int_0^{\frac{\pi}{2}} f(\tan\theta + \cot\theta) \frac{\log(a\tan\theta)}{\sin 2\theta} d\theta \qquad ...(1)$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{f\left(\tan\left(\frac{\pi}{2}-\theta\right) + \cot\left(\frac{\pi}{2}-\theta\right)\log a\left(\tan\left(\frac{\pi}{2}-\theta\right)\right)\right)}{\sin 2\left(\frac{\pi}{2}-\theta\right)} d\theta$$

$$I = 2\int_0^{\frac{\pi}{2}} \frac{f(\cot\theta + \tan\theta)\ \log(a\cot\theta)}{\sin 2\theta} d\theta \qquad ...(2)$$

(1) + (2)

$$2I = 2\int_0^{\frac{\pi}{2}} \frac{f(\tan\theta + \cot\theta)\log(a\tan\theta . a\cot\theta)}{\sin 2\theta} d\theta$$

$$I = \log a^2 \int_0^{\frac{n}{2}} \frac{f(\tan\theta + \cot\theta)}{\sin 2\theta} d\theta$$

$$= \log a \int_0^{\frac{\pi}{2}} \frac{f(\tan\theta + \cot\theta)}{\sin\theta\cos\theta} d\theta$$

after let $\tan\theta = \frac{x}{a}$ $\quad \sec^2\theta = \frac{1}{a}\ dx\ \ d\theta = \frac{\cos^2\theta}{a} dx$

$$= \log a \int_0^{\infty} \frac{f\left(\frac{x}{a} + fa\right) x}{\sin\theta\cos\theta} \frac{\cos^2\theta}{a} dx$$

$$= \log a \int_0^{\infty} \frac{f\left(\left(\frac{x}{a}\right) + \frac{a}{x}\right)}{x} dx$$ **Proved.**

**Q.45**

**Sol.** $y = ax^2 + bx = c$

$y' = 2ax + b$

$y'(2) = 4a + b = 1$

$f(x) = ax^2 + (1 - 4a)x + c$

Now $\int_{2-\pi}^{2+\pi} f(x).\sin\left(\frac{x-2}{2}\right)dt$

let $x-2=t$

$dx = dt$

$$\int_{-x}^{x} f(x+2)\sin\left(\frac{t}{2}\right)dt$$

$$\int_{-\pi}^{\pi}(a(t+2)^2+(1-4a)\ (t+2)+c)\ \sin\frac{t}{2}\,dt$$

$$= \int_{-\pi}^{\pi} at^2+\frac{t}{2}dt + \int_{-\pi}^{\pi} 4a\sin\frac{t}{2}dt + 4a\int_{-\pi}^{\pi} t\sin\frac{t}{2}dt + (1-4a)\int_{-\pi}^{\pi} t\sin\frac{t}{2}dt + 2(1-4a)$$

$$\int_{-\pi}^{\pi}\sin dt + c\int_{-\pi}^{\pi}\sin t\ dt$$

$$= (4a+1-4a)\int_{-\pi}^{\pi} t\sin\frac{t}{2}dt$$

$$= 2\int_{0}^{\pi} t\sin\frac{t}{2}dt$$

$$= 2\left[-2t\cos\frac{t}{2}+4\sin\frac{t}{2}\right]_0^{\pi}$$

$= 2(4) = 8$

**Q.46**

**Sol.** $I = \int x\left(\sqrt{x+\frac{1}{x^2}}\right)\frac{\left[\ell n\, x^2+\ell n\left(1+\frac{1}{x^2}\right)-\ell n x^2\right]^2}{x^4}$

$$= \int\left[x\sqrt{1+\frac{1}{x^2}}\left[\frac{\ell n x^2+\ell n\left(1+\frac{1}{x^2}\right)-2\ell n x}{x^4}\right]dx\right]$$

$$= \int \left[ \frac{1}{x^3} \sqrt{1 + \frac{1}{x^2}} \left[ \ell n \left( 1 + \frac{1}{x^2} \right) \right] \right] dx$$

put $1 + \frac{1}{x^2} = t$

$$\frac{-2}{x^3} dx = dt$$

$$= \left( \frac{-1}{2} \right) \int \sqrt{t} \; \ell nt \; dt$$

$$= \left( \frac{-1}{2} \right) \left[ \int t^{1/2} . \ell nt \; dt \right]$$

$$= -\frac{1}{2} \int \underset{II}{t^{1/2}} . \underset{I}{\ell nt} \; dt \quad \text{(using by parts)}$$

$$= -\frac{1}{2} \left[ \ell nt \int t^{1/2} dt - \int \frac{1}{t} \left( \int t^{1/2} dt \right) dt \right]$$

$$= -\frac{1}{2} \left[ (\ell nt) \frac{t^{3/2}}{3/2} - \int \frac{t^{1/2}}{3/2} \; dt \right]$$

$$= -\frac{1}{3} t^{3/2} \ell nt - \frac{1}{3} \frac{t^{3/2}}{3/2} + c$$

$$\boxed{I = -\frac{1}{3} \left( 1 + \frac{1}{x^2} \right)^{3/2} \ell n \left( 1 + \frac{1}{x^2} \right) - \frac{2}{9} \left( 1 + \frac{1}{x^2} \right)^{3/2} + c}$$

**Q.47**

**Sol.** $I = \int \frac{\tan 2\theta}{\sqrt{\cos^6 \theta + \sin^6 \theta}} \, d\theta$

$$= \int \frac{\tan 2\theta}{\sqrt{\cos^4 \theta (1 - \sin^2 \theta) + \sin^4 \theta (1 - \cos^2 \theta)}} d\theta$$

$$= \int \frac{\tan 2\theta}{\sqrt{\cos^4\theta + \sin^4\theta - \sin^2\theta\cos^2\theta(\cos^2\theta + \sin^2\theta)}}\,d\theta$$

$$= \int \frac{2\tan\theta.\sec^2\theta}{(1-\tan^2\theta)\sec^2\theta.\sqrt{\cos^4\theta + \sin^4\theta - \sin^2\theta\cos^2\theta}}\,d\theta$$

$$= \int \frac{dt}{(1-t)\,\sqrt{t^2 - t + 1}}$$ put $\tan^2\theta = t$

put $1 - t = \dfrac{1}{u}$ $2\tan\theta\sec^2\theta\,d\theta = dt$

$$\text{or } I = \int \frac{du}{u^2.\frac{1}{u}\sqrt{\left(1-\frac{1}{u}\right)^2 - \left(1-\frac{1}{u}\right) + 1}}$$

$$= \int \frac{du}{\frac{u}{u}\sqrt{u^2+1-u}} = \int \frac{du}{\sqrt{\left(u-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}} = \ell n\left[\left(u-\frac{1}{2}\right) + \sqrt{\left(u-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$= \ell n\left[\left(\frac{1}{1-t}-\frac{1}{2}\right) + \sqrt{\left(\frac{1}{1-t}-\frac{1}{2}\right)^2 + \frac{3}{4}}\right] = \ell n\left[\left(\frac{1}{1-\tan^2\theta}-\frac{1}{2}\right) + \sqrt{\left(\frac{1}{1-\tan^2\theta}\frac{1}{2}\right)^2 + \frac{3}{4}} + c\right]$$

**Q.48**

**Sol.** $$I = \int \frac{\cot x\ dx}{(1-\sin x)\ (\sec x + 1)} = \int \frac{\frac{\cos x}{\sin x}}{(1-\sin x)\left(\frac{1}{\cos x}+1\right)}\,dx$$

$$= \int \frac{\cos x(1+\sin x)}{\sin x\cos^2 x\left(\frac{1+\cos x}{\cos x}\right)}\,dx$$

$$= \int \frac{1+\sin x}{\sin x(1+\cos x)}\,dx$$

$$= \int \frac{1+\sin x}{\sin x . 2\cos^2 \frac{x}{2}} dx$$

$$= \frac{1}{2}\int \text{cosec}\, x . \sec^2 \frac{x}{2} dx + \frac{1}{2}\int \sec^2 \frac{x}{2} dx$$

$$= \frac{1}{2}\int \underset{u}{\text{cosec}\, x} . \underset{v}{\sec^2 \frac{x}{2}} dx + \frac{1}{2}\frac{\tan \frac{x}{2}}{\frac{1}{2}} + c$$

$$= \frac{1}{2}\left[\text{cosec}\, x \int \sec^2 \frac{x}{2} dx - \int \left[(-\text{cosex} \cot x)\int \sec^2 \frac{x}{2} dx\right] dx\right] + \tan \frac{x}{2} + c$$

$$= \frac{1}{2}\left[\text{cosecx}. \frac{\tan \frac{x}{2}}{\frac{1}{2}} + \int \text{cosec}\, x \cot x . \frac{\tan \frac{x}{2}}{\frac{1}{2}} dx\right] + \tan \frac{x}{2} + c$$

$$= \text{cosec}\, x . \tan \frac{x}{2} + \int \frac{\cos x}{\sin^2 x} \tan \frac{x}{2} dx + \tan \frac{x}{2} + c$$

$$= \frac{1}{\sin x} \tan \frac{x}{2} + \int \frac{\frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}}{\left(\frac{2\tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}}\right)^2} \tan \frac{x}{2} dx + \tan \frac{x}{2} + c$$

$$= \frac{\left(1+\tan^2 \frac{x}{2}\right)}{2} + \int \frac{\left(1-\tan^2 \frac{x}{2}\right)\left(1+\tan^2 \frac{x}{2}\right)}{4\tan \frac{x}{2}} dx + \tan \frac{x}{2} + c$$

$$= \frac{1}{2}\sec^2\frac{x}{2} + \frac{1}{4}\int \frac{\left(1-\tan^2\frac{x}{2}\right)}{\tan\frac{x}{2}}\sec^2\frac{x}{2}dx + \tan\frac{x}{2} + c$$

put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

$$= \frac{1}{2}\sec^2\frac{x}{2} + \frac{1}{2}\int \frac{1-t^2}{t}dt + \tan\frac{x}{2} + c$$

$$= \frac{1}{2}\sec^2\frac{x}{2} + \frac{1}{2}\ell nt - \frac{1}{4}t^2 + \tan\frac{x}{2} + c$$

$$= \frac{1}{2}\sec^2\frac{x}{2} + \frac{1}{2}\ell n\left(\tan\frac{x}{2}\right) - \frac{1}{4}\tan^2\frac{x}{2} + \tan\frac{x}{2} + c$$

$$\frac{1}{2}\int \frac{1-t^2}{t}\,dt$$

$$= \frac{1}{2}\sec^2\frac{x}{2} + \frac{1}{2}\left[\ell n\left|\tan\frac{x}{2}\right| - \frac{1}{2}\tan^2\frac{x}{2}\right] + \tan\frac{x}{2} + c$$

$$= \frac{1}{2}\sec^2\frac{x}{2} + \frac{1}{2}\ell n\left|\tan\frac{x}{2}\right| - \frac{1}{4}\tan^2\frac{x}{2} + \tan\frac{x}{2} + c$$

$$= \frac{1}{2}\ell n\left|\tan\frac{x}{2}\right| + \tan\frac{x}{2} + \frac{1}{2}\sec^2\frac{x}{2} - \frac{1}{4}\left(\sec^2\frac{x}{2} - 1\right) + c$$

$$\boxed{I = \frac{1}{2}\ell n\left(\tan\frac{x}{2}\right) + \tan\frac{x}{2} + \frac{1}{4}\sec^2\frac{x}{2} + c_1}$$

**Q.49**

**Sol.** $I = \int \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}dx$

$$= \int \frac{e^x(1+1-x^2)}{(1-x)\sqrt{1-x^2}}dx$$

$$= \int e^x \left[ \frac{1}{(1-x)\sqrt{1-x^2}} + \frac{1-x^2}{(1-x)\sqrt{1-x^2}} \right] dx$$

$$= \int e^x \left[ \underbrace{\sqrt{\frac{1+x}{1-x}}}_{f(x)} + \underbrace{\frac{1}{(1-x)\sqrt{1-x^2}}}_{f'(x)} \right] dx$$

$$= e^x . f(x) + c$$

$$\boxed{I = e^x \sqrt{\frac{1+x}{1-x}} + c}$$ **Ans.**

**Q.50**

**Sol.** $I = \int \frac{3x^2+1}{(x^2-1)^3} dx$

$$= \int \frac{3x^2+1-x^2+x^2}{(x^2-1)^3} dx$$

$$= \int \frac{-(x^2-1)}{(x^2-1)^3} dx + \int \frac{4x^2}{(x^2-1)^3} dx$$

$$= \int \left[ \frac{-1}{(x^2-1)^2} + x . \frac{4x}{(x^2-1)^3} \right] dx$$

This is the integral form of

$$\int [f(x) + xf'(x)] dx = xf(x) + c$$

$$= x\left( \frac{-1}{(x^2-1)^2} \right) + c$$

$$\boxed{I = -\frac{x}{(x^2-1)^2} + c}$$ **Ans.**

**Q.51**

**Sol.** $I = \int \frac{(ax^2 - b)\ dx}{x\sqrt{c^2x^2 - (ax^2 + b)^2}}$

diving by $x^2$

$$= \int \frac{\left(a - \frac{b}{x^2}\right) dx}{\sqrt{c^2 - \left(ax + \frac{b}{x}\right)^2}}$$

or $I = \int \frac{dt}{\sqrt{c^2 - t^2}}$ put $ax + \frac{b}{x} = t, \left(a - \frac{b}{x^2}\right) dx = dt$

$I = \sin^{-1}\left(\frac{t}{c}\right) + c$

$$\boxed{I = \sin^{-1}\left(\frac{ax + \frac{b}{x}}{c}\right) + c}$$

**Q.52**

**Sol.** $I = \int \frac{dx}{(x + \sqrt{x(1+x)})^2}\ dx$

$$= \int \frac{1}{x^2\left(1 + \sqrt{1 + \frac{1}{x}}\right)^2} dx$$

put $1 + \frac{1}{x} + t^2$

$-\frac{1}{x^2} dx = 2tdt$

or $I = \int \frac{-2t\ dt}{(1+t)^2} = -\int \frac{2t}{t^2 + 2t + 1}\ dt$

$$= -\left[\int \frac{2t+2}{t^2+2t+1}dt - \int \frac{2}{t^2+2t+1}\,dt\right.$$

$$= -\,\ell n\,(t+1)^2 - 2\int \frac{1}{(t+1)^2}dt$$

$$= -\,2\ell n\,(t+1) + \frac{2}{t+1} + c$$

or $\boxed{I = -2\ell n\left(1+\sqrt{1+\frac{1}{x}}\right) + \frac{2}{1+\sqrt{1+\frac{1}{x}}} + c}$

**Q.53**

**Sol.** $I = \int \frac{x+1}{x(1+xe^x)^2}dx$

$$= \int \frac{(x+1)e^x}{x.e^x(1+xe^x)^2}dx$$

or $I = \int \frac{1}{(t-1)t^2}\,dt$ put $1 + xe^x = t \Rightarrow (x.e^x + e^x.\,1)dx = dt \Rightarrow e^x(1 + x)dx = dt$

$$= \int \frac{(1-t^2)+t^2}{(t-1)t^2}\,dt$$

$$= \int \frac{-(1+t)}{t^2}\,dt + \int \frac{1}{(t-1)}dt$$

$$= -\int \frac{1}{t^2}\,dt - \int \frac{1}{t}dt + \int \frac{1}{(t-1)}dt$$

$$I = \frac{1}{t} - \ell n\,(t) + \ell n\,(t-1) + c$$

$$= \frac{1}{1+xe^x} - \ln(1+xe^x) + \ell n(xe^x) + c$$

$$I = \frac{1}{1+xe^x} + \ell n\left(\frac{xe^x}{1+xe^x}\right) + c$$

**Q.54**

**Sol.** Let $f(x) = ax^2 + bx + 1$

$$I = \int \frac{f(x)dx}{x^2(x+1)^3}$$

$$= \int \frac{ax^2 + bx + 1}{x^2(x+1)^3}\,dx$$

or $\dfrac{ax^2+bx+1}{x^2(x+1)^3} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{(x+1)} + \dfrac{D}{(x+1)^2} + \dfrac{E}{(x+1)^3}$

**Q.55**

**Sol.** $f'(x) = \dfrac{1}{1+x^2} + \dfrac{1}{2}\left(\dfrac{1}{1+x} - \dfrac{-1}{1-x}\right) = \dfrac{2}{1-x^4}$

$$\int \frac{1}{2} f'(x)\,d(x^4) = \int \frac{1}{2}\cdot\frac{2}{1-x^4}.4x^3dx = \int \frac{-1}{t}dt$$

(where $t = 1 - x^4, dt = -4x^3dx$ )

$$= -\ell n\, t + c = -\ell n\left|1 - x^4\right| + c$$

**Q.56**

**Sol.** $= \displaystyle\int \frac{x(1+x)}{e^{2x}\left(1 + \frac{x}{e^x} + \frac{1}{e^x}\right)^2}dx$

$$= \int \frac{x(1+x)e^{-2x}}{[1+(1+x)e^{-x}]^2} dx$$

$$\text{or } \; I = \int \frac{x(1+x)e^{-x}.xe^{-x}}{(1+(1+x)e^{-x})^2} dx$$

put $1 + (x + 1)e^{-x} = t$

$[0 + e^{-x} 1 + x (-e^{-x}) + e^{-x} (-1)]dx = dt$

$-xe^{-x} dx = dt$

$$\text{or } I = -\int \frac{(t-1)}{t^2} dt$$

$$= \int \frac{1}{t^2} dt - \int \frac{1}{t} dt$$

$$\text{or } \; I = -\frac{1}{t} - \ell nt + c$$

$$\text{or } \; \boxed{I = -\frac{1}{1+(1+x)e^{-x}} - \ell n\,|1+(1+x)e^{-x}| + c}$$

**Q.57**

**Sol.** $I = \int \frac{e^{\cos x}(x \sin^3 x + \cos x)}{\sin^2 x} dx$

$$= \int e^{\cos x}(x \sin x + \cot x \operatorname{cosec} x) dx$$

$$\text{or } I = \int \underset{I}{x}.e^{\cos x} \underset{II}{\sin x} \, dx + \int \underset{I}{e^{\cos x}} \underset{II}{\operatorname{cosec}} \cot x \, dx$$

$I = I_1 + I_2$

$$I_1 = \int \underset{I}{x} \; e^{\cos x}.\underset{II}{\sin x}\, dx = x \int e^{\cos x} \sin x dx - \int 1.\left(\int e^{\cos x} \sin x \; dx\right) dx$$

$$= -\; xe^{\cos x} + \int 1.e^{\cos x} dx + c$$

$$I_2 = \int \frac{e^{\cos x}}{I} . \frac{\operatorname{cosec} x \cot x}{II} dx$$

$$= e^{\cos x} \int \operatorname{cosec} x \cot x \; dx - \int (e^{\cos x}(-\sin x) \int \operatorname{cosec} x \; \cot x \, dx) dx$$

$$= e^{\cos x}(-\text{cosec}x)+\int e^{\cos x}.\sin x(-\text{cosec}\,x)dx$$

$$= -e^{\cos x}\text{ cosec } x - \int e^{\cos x}dx + c$$

$$\therefore\ I = I_1 + I_2 = -xe^{\cos x} + \int e^{\cos x}dx - e^{\cos x}\text{cosec } x - \int e^{\cos x}dx + c$$

$$= -e^{\cos x}(x + \text{cosec } x) + c$$

**Q.58**

**Sol.** $I = \int \frac{5x^4+4x^5}{(x^5+x+1)^2}dx$

$$= \int \frac{5x^4+1}{(x^5+x+1)^2}dx + \int \frac{4x^5-1}{(x^5+x+1)^2}dx$$

$I = I_1 + I_2$

$I_1$

$\Rightarrow$ put $x^5 + x + 1 = t$

$$= \int \frac{1}{t^2}dt$$

$$= -\frac{1}{t}+c$$

$$= -\frac{1}{x^5+x+1}+c$$

$I_2$ (dividing by $x^2$)

$$\Rightarrow \int \frac{4x^3-\frac{1}{x^2}}{\left(x^4+1+\frac{1}{x}\right)^2}dx$$

put $x^4+\frac{1}{x}+1=t$

$$\left(4x^3-\frac{1}{x^2}\right)dx = dt$$

$$= \int \frac{1}{t^2}dt$$

$$= -\frac{1}{x^4+\frac{1}{x}+1}+c$$

$$= -\frac{-x}{x^5+x+1}+c$$

or $I = I_1 + I_2$

or $\boxed{I = -\frac{(x+1)}{x^5+x+1} + c}$

**Q.60**

**Sol.** $I = \int \frac{\cos^2 x}{1+\tan x} dx = \frac{1}{2}\int \frac{2\cos^3 x}{\sin x + \cos x} dx$

$$= \frac{1}{2}\int \frac{\cos^3 x - \sin^3 x + \cos^3 x + \sin^3 x}{(\sin x + \cos x)} dx$$

$$= \frac{1}{2}\int \frac{(\cos x - \sin x)\ (1+\sin x \cos x)}{\sin x + \cos x} dx + \frac{1}{2}\int \frac{(\cos x + \sin x)}{(\cos x + \sin x)}(1 - \sin x \cos x) dx$$

= put sinx + cosx = t

(cos x – sin x)dx = dt

$$= \frac{1}{2}\int \frac{\left[1+\frac{1}{2}(t^2-1)\right]}{t} dt + \frac{1}{2}\int \left(1 - \frac{1}{2}\sin 2x\right) dx$$

$$= \frac{1}{2}[\log t] + \frac{1}{4}\left[\frac{t^2}{2} - \log t\right] + \frac{1}{2}x - \frac{1}{4}\frac{(-\cos 2x)}{2} + c$$

$$= \frac{(\sin x + \cos x)^2}{8} + \frac{1}{4}\log(\sin x + \cos x) + \frac{x}{2} + \frac{1}{8}\cos 2x + c$$

$$= \frac{1}{8} + \frac{\sin 2x}{8} + \frac{1}{4}\log(\sin x + \cos x) + \frac{x}{2} + \frac{\cos 2x}{8} + c$$

or $\boxed{I = \frac{(\sin 2x + \cos 2x)}{8} + \frac{1}{4}\log(\sin x + \cos x) + \frac{x}{2} + c}$

**Q.61**

**Sol.** $I = \int \frac{x^3+x+1}{x^4+x^2+1} dx$

$$=\int \frac{x^3+x+1}{x^4+x^2+1+x^2-x^2}dx$$

$$=\int \frac{x^3+x+1}{(x^2+1)^2-x^2}dx$$

or $I=\int \frac{x^3+x+1}{(x^2+x+1)\,(x^2-x+1)}dx$

Now $\frac{x^3+x+1}{(x^2+1-x)\,(x^2+1+x)}=\frac{Ax+B}{(x^2+1-x)}+\frac{Cx+D}{x^2+1+x}$

on comparing, A = 0, B = $\frac{1}{2}$ , C = 1, $D=\frac{1}{2}$

$$=\frac{x+\frac{1}{2}}{x^2+1+x}+\frac{\frac{1}{2}}{x^2+1-x}$$

or I = $\frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx+\frac{1}{2}\int \frac{1}{x^2-x+1+\frac{1}{4}-\frac{1}{4}}dx$

$$=\frac{1}{2}\log(x^2+x+1)+\frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

or $\boxed{I=\frac{1}{2}\log(x^2+x+1)+\frac{1}{2}.\frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c}$

**Q.62**

**Sol.** $I=\int (\sin x)^{-11/3}(\cos x)^{-1/3}dx$

$$= \int \frac{(\sin x)^{1/3}}{\sin^4 x (\cos x)^{1/3}} dx$$

$$= \int \frac{\text{cosec}^4 x}{(\cot x)^{1/3}} dx$$

$$= \int \frac{(1+\cot^2 x)\,\text{cosec}^2 x}{(\cot x)^{1/3}} dx$$

$$= -\int \frac{1+t^2}{t^{1/3}} dt$$ put cot x = t $\Rightarrow$ cosec$^2$x dx = – dt

$$= -\left[\frac{t^{-1/3+1}}{\frac{-1}{3}+1} + \frac{t^{2-\frac{1}{3}+1}}{2-\frac{1}{3}+1}\right] + c$$

$$= -\left[\frac{3}{2}t^{2/3} + \frac{3}{8}t^{8/3}\right] + c$$

or $\boxed{I = -\left[\frac{3}{2}(\cot x)^{2/3} + \frac{3}{8}(\cot x)^{8/3}\right] + c}$

**Q.63**

**Sol.** $I = \int \frac{dx}{\sqrt{\sin^3 x \sin(x+\alpha)}} dx$

$$= \int \frac{1}{\sqrt{\sin^3 x[\sin x \cos\alpha + \cos x \sin\alpha]}} dx$$

$$= \int \frac{1}{\sqrt{\sin^4 x[\cos\alpha + \cot x \sin\alpha]}} dx$$

$$= \int \frac{\text{cosec}^2 x}{\sqrt{\cot x.\sin\alpha + \cos\alpha}} dx$$

put sinα. cotx + cosα = $t^2 \Rightarrow$ – sinα cosec$^2$x dx = 2t dt or $I = \int \frac{-1}{\sin\alpha} \frac{2t}{t} dt$

$= -\frac{2}{\sin\alpha} . \int 1.dt$

$= -\frac{2}{\sin\alpha} t + c$

$$\boxed{I = -\frac{2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}} + c}$$

**Q.64**

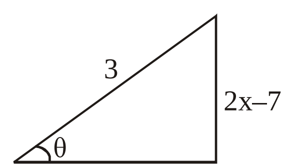
**Sol.** $I = \int \frac{x}{(7x-10-x^2)^{3/2}} dx$

$= \int \frac{x}{\left(\sqrt{\frac{1}{4}[9-(2x-7)^2]}\right)^3} dx$

put 2x – 7 = 3 sin θ ⇒ 2dx = 3cos θ dθ

$= \frac{3}{4}\int \frac{3\sin\theta + 7}{\left(\sqrt{\frac{1}{4}(9-9\sin^2\theta)}\right)^3} d\theta$

$\sin\theta = \frac{2x-7}{3}$




$= \frac{3}{4}\int \frac{3\sin\theta + 7}{\left(\frac{3}{2}\cos\theta\right)^3} d\theta$

$\cos\theta = \frac{\sqrt{9-(2x-7)^2}}{3}$

$= \frac{3}{4} \times \frac{8}{27}\int \frac{3\sin\theta + 7}{\cos^3\theta} d\theta$

$\tan\theta = \frac{2x-7}{\sqrt{9-(2x-7)^2}}$

$= \frac{2}{9}\int \frac{3\sin\theta}{\cos^3\theta} d\theta + \frac{2}{9}\int \frac{7}{\cos^3\theta} d\theta$

$$= \frac{2}{3}\int \frac{\sin\theta}{\cos^3\theta}d\theta + \frac{14}{9}\int \sec^3\theta\, d\theta$$

put $\cos\theta = t \Rightarrow -\sin\theta\, d\theta = dt$ (using by part method)

$$= -\frac{2}{3}\int t^{-3}dt + \frac{14}{9}\sec\theta(\tan\theta - 1)$$

$$= \frac{1}{3}.\frac{1}{t^2} + \frac{14}{9}.\frac{3}{\sqrt{9-(2x-7)^2}}\left(\frac{2x-7}{\sqrt{9-(2x-7)^2}} - 1\right)$$

**Q.65**

**Sol.** $I = \int \frac{dx}{\sec x + \operatorname{cosec} x}$

$$= \int \frac{1}{2} \times \frac{2\sin x \cos x}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x)^2 - 1}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2}\int (\sin x + \cos x)dx - \frac{1}{2}\int \frac{1}{\sin x + \cos x}dx$$

$$= \frac{1}{2}(-\cos x + \sin x) - \frac{1}{2}\int \frac{1}{\sqrt{2}\left[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right]}\, dx$$

$$= \frac{1}{2}(-\cos x + \sin x) - \frac{1}{2\sqrt{2}}\int \frac{1}{\sin\left(x + \frac{\pi}{4}\right)}dx$$

$$= \frac{1}{2}(-\cos x + \sin x) - \frac{1}{2\sqrt{2}}\int \operatorname{cosec}\left(x + \frac{\pi}{4}\right) dx$$

$$\boxed{I = \frac{1}{2}(-\cos x + \sin x) - \frac{1}{2\sqrt{2}}\log\tan\left(\frac{x}{2} + \frac{\pi}{8}\right) + c}$$

**Q.66**

**Sol.** $I = \int \frac{dx}{\sin x + \sec x} dx$

$$= \int \frac{\cos x}{1 + \sin x \cos x} dx$$

$$= \int \frac{2\cos x}{2 + 2\sin x \cos x} dx$$

$$= \int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{2 + \sin 2x} dx$$

$$= \int \frac{\cos x + \sin x}{2 + \sin 2x} dx + \int \frac{\cos x - \sin x}{2 + \sin 2x} dx$$

$$= \int \frac{\cos x + \sin x}{2 + \{1 - (\sin x - \cos x)^2\}} dx + \int \frac{\cos x - \sin x}{2 + \{(\sin x + \cos x)^2 - 1\}} dx$$

put $\sin x - \cos x = u$ put $\sin x + \cos x = v$

$(\cos x + \sin x)dx = du$ $(\cos x - \sin x)dx = du$

$$= \int \frac{1}{(\sqrt{3})^2 - u^2} du + \int \frac{dv}{1^2 + v^2}$$

$$= \frac{1}{2\sqrt{3}} \log\left|\frac{\sqrt{3} + u}{\sqrt{3} - u}\right| + \tan^{-1} v + c$$

$$\boxed{I = \frac{1}{2\sqrt{3}} \log\left|\frac{\sqrt{3} + \sin x - \cos x}{\sqrt{3} - \sin x + \cos x}\right| + \tan^{-1}(\sin x + \cos x) + c}$$

**Q.67**

**Sol.** $I = \int \frac{x^2 + 1}{x^4 - 2x^2 \cos\alpha + 1} dx$

divide by $x^2$ on $N^r$ and $D^r$

$$= \int \frac{\left(1 + \frac{1}{x^2}\right)}{x^2 + \frac{1}{x^2} - 2\cos\alpha + 2 - 2} dx$$

$$= \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2 + 2(1-\cos\alpha)} dx$$

$$= \int \frac{dt}{t^2 + \left(2\sin\frac{\alpha}{2}\right)^2}$$

put $x - \frac{1}{x} = t$

$$\left(1+\frac{1}{x^2}\right) dx = dt$$

or $\boxed{I = \frac{1}{2}\left(\text{cosec}\,\frac{\alpha}{2}\right)\tan^{-1}\left(\frac{x-\frac{1}{x}}{2\sin\frac{\alpha}{2}}\right) + c}$

**Q.68**

**Sol.** $I = \int \frac{\cos x - \sin x}{7 - 9\sin 2x} dx$

or $I = \int \frac{dt}{7 - 9(t^2 - 1)}$

Let $\boxed{\sin x + \cos x = t} \Rightarrow (\cos x - \sin x)dx = dt$

$$= \int \frac{dt}{4^2 - (3t)^2}$$

$\because \sin^2 x + \cos^2 x + 2\sin x\cos x = t^2$

or $\sin^2 x + \cos^2 x + 2\sin x \cos x = t^2$

$$= \int \frac{dt}{4^2 - (3t)^2}$$

or $1 + \sin 2x = t^2$

or $\boxed{\sin 2x = t^2 - 1}$

$$= \frac{1}{2.4}.\frac{1}{3}\ell n\left|\frac{4+3t}{4-3t}\right| + c$$

or $\boxed{I = \frac{1}{24}\ell n\left|\frac{4+3(\sin x+\cos x)}{4-3(\sin x+\cos x)}\right| + c}$

**Q.69**

**Sol.** $I = \int \frac{\sqrt{\cot x} - \sqrt{\tan x}}{1+3\sin 2x}\, dx$

$= \int \frac{(\cos x - \sin x)}{\sqrt{\sin x \cos x}(1+3\sin 2x)}\, dx$

$= \int \frac{1}{\sqrt{\frac{t^2-1}{2}}(3t^2-2)} dt$ put $\sin x + \cos x = t$

$(\cos x - \sin x)\, dx = dt$

$= \sqrt{2}\int \frac{1}{(3t^2-2)\sqrt{t^2-1}} dt$ & $1 + \sin 2x = t^2 \Rightarrow \sin 2x = t^2 - 1$

put $t = \frac{1}{u} \Rightarrow dt = -\frac{1}{u^2} du$

$\sqrt{2}\int \frac{-du}{u^2\left(\frac{3}{u^2}-2\right)\sqrt{\frac{1}{u^2}-1}} = -\sqrt{2}\int \frac{u\,du}{(3-2u^2)\sqrt{1-u^2}}$

**Q.70**

**Sol.** $I = \int \frac{4x^5 - 7x^4 + 8x^3 - 2x^2 + 4x - 7}{x^2(x^2+1)^2}\, dx$

$= \int \frac{4x^5 - 7x^4 + 8x^3 - 2x^2 + 4x - 7}{x^2(x^4+2x^2+1)}\, dx$

$= \int \frac{4x(x^4+2x^2+1) - 7(x^4+1+2x^2) + 12x^2}{x^2(x^4+2x^2+1)}\, dx$

$= \int \frac{4}{x}\, dx - \int \frac{7}{x^2} dx + \int \frac{12}{(x^2+1)^2}\, dx$ put $x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

$$= 4\log x - 7\left(-\frac{1}{x}\right) + \int \frac{12\sec^2\theta d\theta}{(1+\tan^2\theta)^2}$$

$$= 4\log x + \frac{7}{x} + 12\int \frac{1}{\sec^2\theta} d\theta$$

$$= 4\ \ell n|\ x\ | + \frac{7}{x} + 12\int \cos^2\theta\, d\theta$$

$$= 4\ \ell n|\ x\ | + \frac{7}{x} + 12\int \left(\frac{\cos 2\theta + 1}{2}\right) d\theta$$

$$= 4\ell n\ |\ x\ | + \frac{7}{x} + 6\theta + 6\ \frac{\sin^2\theta}{2} + c$$

$$= 4\ell n|x| + \frac{7}{x} + 6\tan^{-1}x + 3\sin(2\tan^{-1}x) + c$$

$$= 4\ell n|\ x\ | + \frac{7}{x} + 6\tan^{-1}x + 3\left[\frac{2\tan(\tan^{-1}x)}{1+[\tan(\tan^{-1}x)]^2}\right] + c$$

$$\boxed{I = 4\ell n\ |\ x\ | + \frac{7}{x} + 6\tan^{-1}x + \frac{6x}{1+x^2} + c}$$

**Q.71**

**Sol.** $I = \int \sqrt{\frac{(1-\sin x)\ (2-\sin x)}{(1+\sin x)\ (2+\sin x)}}\, dx$

$$= \int \frac{\cos x}{(1+\sin x)} \frac{\sqrt{4-\sin^2 x}}{2+\sin x} dx$$

$$= \int \frac{\sqrt{4-(t-1)^2}}{t(1+t)} dt$$ put $1 + \sin x = t \Rightarrow \cos x\, dx = dt$

$$= \int \frac{\sqrt{3+2t-t^2}}{t(1+t)} dt = \int \frac{-(t-3)\ (t+1)}{t(1+1)\sqrt{3+2t-t^2}} dt$$

$$\text{or } I = \int \frac{(3-t)}{t\sqrt{3+2t-t^2}}\,dt$$

$$\text{or } I = 3\int \frac{1}{t\sqrt{3+2t-t^2}} - \int \frac{1}{\sqrt{3+2t-t^2}}\,dt$$

$$\text{put } t = \frac{1}{v}$$

$$dt = -\frac{1}{v^2}\,dv$$

$$\text{or } I = 3\int \frac{-1}{v^2}.\frac{dv}{\frac{1}{v}\sqrt{3+\frac{2}{v}-\frac{1}{v^2}}} - \int \frac{1}{\sqrt{(2)^2-(t-1)^2}}\,dt$$

$$= -3\int \frac{dv}{\sqrt{3v^2+2v-1}} - \int \frac{1}{\sqrt{(2)^2-(t-1)^2}}\,dt$$

$$= -3\int \frac{dv}{\sqrt{\left(\sqrt{3}v-\frac{1}{\sqrt{3}}\right)^2-\left(\frac{2}{\sqrt{3}}\right)^2}} - \int \frac{1}{\sqrt{(2)^2-(t-1)^2}}\,dt$$

$$= -3\sqrt{3}\int \frac{dv}{\sqrt{(3v-1)^2-(2)^2}} - \int \frac{1}{\sqrt{(2)^2-(t-1)^2}}\,dt$$

$$= -\frac{3\sqrt{3}}{3}\log\left[(3v-1)+\sqrt{(3v-1)^2-4}\right] - \sin^{-1}\left(\frac{t-1}{2}\right) + c$$

$$= -\sqrt{3}\log\left[\left(\frac{3}{1+\sin x}-1\right)+\sqrt{\left(\frac{3}{1+\sin x}-1\right)^2-4}\right] - \sin^{-1}\left(\frac{\sin x}{2}\right) + c$$

**Q.72**

**Sol.** $I = \int \frac{dx}{\cos^3 x - \sin^3 x}$

$$= \int \frac{1}{(\cos x - \sin x)\left(1 + \frac{\sin 2x}{2}\right)} dx$$

$$= \int \frac{(\cos x - \sin x)}{(\cos x - \sin x)^2 \left(1 + \frac{\sin 2x}{2}\right)} dx$$

$$= \int \frac{(\cos x - \sin x)}{(1 - \sin 2x)\left(1 + \frac{\sin 2x}{2}\right)} dx$$

$$= \int \frac{(\cos x - \sin x)}{(1 - \sin 2x)\left(1 + \frac{\sin 2x}{2}\right)} dx$$

$$= 2\int \frac{(\cos x - \sin x)}{[2 - (\sin x + \cos x)^2]\,[1 + (\sin x + \cos x)^2]} dx$$

put $\sin x + \cos x = t \Rightarrow (\cos x - \sin x)dx = dt$

$$= 2\int \frac{dt}{(2 - t^2)\,(1 + t^2)}$$

$$= \frac{2}{3}\int \left(\frac{1}{1 + t^2} + \frac{1}{2 - t^2}\right) dt$$

$$= \frac{2}{3}\int \frac{1}{1 + t^2} dt + \frac{2}{3}\int \frac{1}{2 - t^2} dt$$

$$= \frac{2}{3}\tan^{-1} t + \frac{2}{3}.\frac{1}{2\sqrt{2}} \log\left|\frac{\sqrt{2} + t}{\sqrt{2} - t}\right| + c$$

or $\boxed{I = \frac{2}{3}\tan^{-1}(\sin x + \cos x) + \frac{1}{3\sqrt{2}} \log\left|\frac{\sqrt{2} + (\sin x + \cos x)}{\sqrt{2} - (\sin x + \cos x)}\right| + c}$

**Q.73**

**Sol.** $I = \int \frac{dx}{(x - \alpha)\sqrt{(x - \alpha)(x - \beta)}}$

put $x-\alpha = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}dt$

$$= \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1}{t}\left[\frac{1}{t}+\alpha-\beta\right]}}$$

$$= -\int \frac{dt}{\sqrt{1+(\alpha-\beta)t}}$$

$$= -\int \frac{1}{u}.\frac{2u\,du}{(\alpha-\beta)}$$

put $1+(\alpha-\beta)\,t = u^2 \Rightarrow (\alpha-\beta)dt = 2udu \Rightarrow dt = \frac{2u}{(\alpha-\beta)}du$

$$= -\frac{2}{(\alpha-\beta)}.u + c$$

$$= -\frac{2}{(\alpha-\beta)}\sqrt{1+(\alpha-\beta)t} + c$$

or $\boxed{I = -\frac{2}{(\alpha-\beta)}\sqrt{1+\frac{(\alpha-\beta)}{(x-\alpha)}} + c}$

or $\boxed{I = -\frac{2}{(\alpha-\beta)}\sqrt{\frac{(x-\beta)}{(x-\alpha)}} + c}$

**Q.74**

**Sol.** $I = \int \frac{\sqrt{\cos 2x}}{\sin x}\,dx$

$$= \int \sqrt{\frac{\cos^2 x - \sin^2 x}{\sin^2 x}}\,dx$$

$$= \int \sqrt{\cot^2 x - 1}\,dx$$

putting $\cot x = \sec\theta \Rightarrow -\mathrm{cosec}^2 x\,dx = \sec\theta\tan\theta\,d\theta$

or $I = \int \sqrt{\sec^2\theta - 1} \times \frac{\sec\theta\tan\theta}{-\text{cosec}^2 x} d\theta$

$\cot x = \sec\theta \Rightarrow 1 + \cot^2 x = 1 + \sec^2\theta \Rightarrow \text{cosec}^2 x = 1 + \sec^2\theta$

$$= -\int \frac{\sec\theta\tan^2\theta}{1+\sec^2\theta}\, d\theta$$

$$= \int \frac{\sec^2\theta}{\cos\theta + \cos^3\theta}\, d\theta$$

$$= -\int \frac{1-\cos^2\theta}{\cos\theta(1+\cos^2\theta)}\, d\theta$$

$$= -\int \sec\theta\, d\theta + 2\int \frac{\cos\theta}{1+\cos^2\theta} d\theta$$

$$= -\int \sec\theta\, d\theta + 2\int \frac{\cos\theta}{2-\sin^2\theta} d\theta$$

put $\sin\theta = t \Rightarrow \cos\theta\, d\theta = dt$

$$= -\log|\sec\theta + \tan\theta| + 2 \times \frac{1}{2\sqrt{2}} \log\left|\frac{\sqrt{2}+\sin\theta}{\sqrt{2}-\sin\theta}\right| + c$$

$$= -\log|\sec\theta + \tan\theta| + \frac{1}{\sqrt{2}} \log\left|\frac{\sqrt{2}+\sqrt{1-\cos^2\theta}}{\sqrt{2}-\sqrt{1-\cos^2\theta}}\right| + c$$

or $\boxed{I = -\log\left|\cot x + \sqrt{\cot^2 x - 1}\right| + \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^x}}\right| + c}$

**Q.75**

**Sol.** $I = \int \frac{\sqrt{\sin^4 x + \cos^4 x}}{\sin^3 x \cos x} dx$

$$= \int \frac{\sin^2 x\sqrt{1+\cot^4 x}}{\sin x.\sin^3 x.\frac{\cos x}{\sin x}} dx$$

$$= \int \frac{\sqrt{1+(\cot^2 x)^2}}{\sin^2 x.\cot x} dx$$

put $\cot^2 x = t \Rightarrow 2\cot x. (-\text{cosec}^2 x)\, dx = dt$

$$= \int \frac{-\sqrt{1+t^2}}{2\cot x.\cot x} dt$$

$$= -\frac{1}{2}\int \frac{\sqrt{1+t^2}}{t}\, dt$$

$$= -\frac{1}{2}\int \frac{(1+t^2)}{t\sqrt{1+t^2}} dt$$

$$= -\frac{1}{2}\left[\int \frac{1}{t\sqrt{1+t^2}} dt + \int \frac{t}{\sqrt{1+t^2}} dt\right]$$

put $t = \frac{1}{u}$ put $1 + t^2 = v^2$

$2t\, dt = 2v\, dv$

$$= -\frac{1}{2}\left[\int \frac{u^2}{\sqrt{1+u^2}} du + \int \frac{v\,dv}{v}\right]$$

$$= -\frac{1}{2}\left[\int \sqrt{1+u^2} dx - \int \frac{1}{\sqrt{1+u^2}} du + v\right]$$

$$= -\frac{1}{2}\left[\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ell n(u+\sqrt{u^2+1}) - \ell n(u+\sqrt{u^2+1}) + \sqrt{1+t^2}\right]$$

$$= \left(-\frac{u}{4}\sqrt{1+u^2} + \frac{1}{4}\ell n(4+\sqrt{u^2+1}) - \frac{\sqrt{1+t^2}}{2}\right)$$

$$= -\frac{\tan^2 x}{4}.\sec x + \frac{1}{u}\ell n(\tan^2 x + \sqrt{1+\tan^4 x}) - \frac{1}{2}\sec x + c$$

**Q.76**

**Sol.** $I = \int \frac{1-(\cot x)^{2008}}{\tan x + (\cot x)^{2009}} dx = \frac{1}{k} \ell n |\sin^k x + \cos^k x| + C$

$$\Rightarrow \int \frac{1-\left(\frac{\cos x}{\sin x}\right)^{2008}}{\frac{\sin x}{\cos x}+\left(\frac{\cos x}{\sin x}\right)^{2009}} dx$$

$$\Rightarrow \int \frac{\sin^{2008} x - \cos^{2008} x}{\sin^{2008} x \frac{(\sin^{2010} x - \cos^{2010} x)}{\sin^{2009} x \cos x}} dx$$

$$= \int \frac{(\sin^{2008} x - \cos^{2008} x)\sin x \cos x}{\sin^{2010} x + \cos^{2010} x} dx$$

put $\sin^{2010} x + \cos^{2010} x = t \Rightarrow [(2010)\sin^{2009} x. \cos x + 2010 \cos^{2009} x (-\sin x)] dx = dt$

$\Rightarrow (2010) \sin x. \cos x [\sin^{2008} x - \cos^{2008} x]dx = dt$

$$= \frac{1}{2010} \int \frac{1}{t} dt$$

$$= \frac{1}{2010} \log |t| + c$$

or $\Rightarrow \frac{1}{2010} \log |\sin^{2010} x + \cos^{2010} x| + c = \frac{1}{k} \log |\sin^{2010} x + \cos^{2010} x| + c$

$\boxed{k = 2010}$ **Ans.**

**Q.77**

**Sol.** $I = \int \cos 2\theta . \ell n \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} d\theta$

$$= \frac{1}{2} \int \cos 2\theta \log \left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)^2 d\theta$$

$$= \frac{1}{2} \int \cos 2\theta \log \left(\frac{1 + \sin 2\theta}{1 - \sin 2\theta}\right) d\theta$$

$$= \frac{1}{4} \int \log \left(\frac{1 + \sin 2\theta}{1 - \sin 2\theta}\right) .2 \cos 2\theta \, d\theta$$

put $\sin 2\theta = t$

$2\cos 2\theta \, d\theta = dt$

$= \frac{1}{4}\int \log\left(\frac{1+t}{1-t}\right) dt$

$= \frac{1}{4}\left[\int \log(1+t)dt - \int \log(1-t)dt\right]$

$= \frac{1}{4}$ [t log (1 + t) – t + log (1 + t) – t log (1 – t) + t + log (1 – t)]

$= \frac{1}{4}\left[t\log\left(\frac{1+t}{1-t}\right) + \log(1-t^2)\right] + c$

$= \frac{1}{4}\left[\sin 2\theta \log\left(\frac{1+\sin 2\theta}{1-\sin\theta}\right) - \frac{1}{2}\ell n(\sec^2 2\theta) + c\right]$

or $\boxed{I = \frac{1}{2}(\sin 2\theta)\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) - \frac{1}{2}\ell n(\sec 2\theta) + c}$

**Q.78**

**Sol.** $I = \int \frac{x^2}{(x\cos x - \sin x)\,(x\sin x + \cos x)}\, dx$

$I = \int\left[\frac{x\cos x}{(x\sin x + \cos x)} + \frac{x\sin x}{x\cos x - \sin x}\right] dx$

$= \int \frac{x\cos x}{x\sin x + \cos x}\, dx \quad + \quad \int \frac{x\sin x}{x\cos x - \sin x} dx$

$I_1$ $I_2$

put x sin x + cos x = t | put x cosx – sin x = t

(x cos x + sin x – sin x) dx = dt | (–x sin x + cos x – cos x) dx = dt

x cos x dx = dt | – x sin x dx = dt

or $I_1 = \int \frac{1}{t} dt$ $\qquad$ $I_2 = -\int \frac{1}{t} dt$

$= \ell n\,|x \sin x + \cos x| + c$ $\qquad$ $= -\ell n|x \cos x - \sin x| + c$

or $I = I_1 + I_2$

$$\boxed{I = \ell n \left| \frac{x\sin x + \cos x}{x \cos x - \sin x} \right| + c}$$ **Ans**