Q.1

Sol. $I = \int \frac{dx}{\cot\frac{x}{2}.\cot\frac{x}{3}.\cot\frac{x}{6}}$

$$= \int \tan\frac{x}{2}.\tan\frac{x}{3}.\tan\frac{x}{6}\,dx$$

$$\because\ \frac{x}{2}-\frac{x}{3}=\frac{x}{6}$$

$$\tan\left(\frac{x}{2}-\frac{x}{3}\right)=\tan\frac{x}{6}$$

$$\tan\left(\frac{x}{2}-\frac{x}{3}\right)=\tan\frac{x}{6}$$

or $\frac{\tan\frac{x}{2}-\tan\frac{x}{3}}{1+\tan\frac{x}{2}\tan\frac{x}{3}}=\tan\frac{x}{6}$

or $\boxed{\tan\frac{x}{2}-\tan\frac{x}{3}-\tan\frac{x}{6}=\tan\frac{x}{2}\tan\frac{x}{3}\tan\frac{x}{6}}$

or $I = \int\left(\tan\frac{x}{2}-\tan\frac{x}{3}-\tan\frac{x}{6}\right)dx$

$$= \frac{\ell n\left(\sec\frac{x}{2}\right)}{\frac{1}{2}}-\frac{\ell n\left(\sec\frac{x}{3}\right)}{\frac{1}{3}}-\frac{\ell n\left(\sec\frac{x}{6}\right)}{\frac{1}{6}}+c$$

or $\boxed{I = 2\ell n\left(\sec\frac{x}{2}\right)-3\ell n\left(\sec\frac{x}{3}\right)-6\ell n\left(\sec\frac{x}{6}\right)+c}$

$$= \int \tan\frac{x}{2}.\frac{\sec^2\frac{x}{2}}{\sqrt{\left(2-\sec^2\frac{x}{2}\right)\left(2+\sec^2\frac{x}{2}\right)}}dx$$

$$= \int \frac{\tan\frac{x}{2}\sec^2\frac{x}{2}}{\sqrt{4-\sec^4\frac{x}{2}}}dx$$

put $\boxed{\sec^2\frac{x}{2} = t}$

or $2\sec\frac{x}{2} \times \sec\frac{x}{2}\tan\frac{x}{2} \times \frac{1}{2}dx = dt$

or $\boxed{\sec^2\frac{x}{2}\tan\frac{x}{2}dx = dt}$

$$= \int \frac{dt}{\sqrt{4-t^2}}$$

$$= \sin^{-1}\left(\frac{t}{2}\right) + c$$

or $\boxed{I = \sin^{-1}\left(\frac{1}{2}\sec^2\frac{x}{2}\right) + c}$ **Ans**

Q.3

Sol. $$\int \frac{\ell n\left(\ell n\left(\frac{1+x}{1-x}\right)\right)}{1-x^2}dx$$

put $\ell n\left(\frac{1+x}{1-x}\right) = t$

$$\left(\frac{1-x}{1+x}\right) \times \frac{(1-x)-(1+x)(-1)}{(1-x)^2}dx = dt$$

$$= \int 1.dt + \int \frac{1}{t^2} dt$$

$$= t - \frac{1}{t} + c$$

or $\boxed{I = \left(\frac{x}{e}\right)^x - \left(\frac{e}{x}\right)^x + c}$

Q.5

Sol. $I = \int \sqrt{\frac{\sin(x-a)}{\sin(x+a)}} dx$

$$= \int \sqrt{\frac{\sin(x-a) \times \sin(x-a)}{\sin(x+a)\sin(x-a)}} dx$$

$$= \int \frac{\sin(x-a)}{\sqrt{\sin^2 x - \sin^2 a}} dx$$

$$= \int \frac{\sin x \cos a}{\sqrt{1-\cos^2 x - \sin^2 a}} dx - \int \frac{\cos x \sin a}{\sqrt{\sin^2 x - \sin^2 a}} dx$$

$$= \int \frac{\sin x \cos a}{\sqrt{(1-\sin^2 a) - \cos^2 x}} - \int \frac{\cos x \sin a}{\sqrt{\sin^2 x - \sin^2 a}} dx$$

$I = I_1 - I_2$

$I_1 = \int \frac{\sin x \cos a}{\sqrt{\cos^2 a - \cos^2 x}} dx$

put $\cos x = u$

$-\sin x\, dx = du$

$= -\cos a \int \frac{du}{\sqrt{\cos^2 a - u^2}}$

$= -\cos a \sin^{-1}\left(\frac{\cos x}{\cos a}\right)$

$I_2 = \int \frac{\cos x \sin a}{\sqrt{\sin^2 x - \sin^2 a}} dx$

put $\sin x = v$

$\cos x\, dx = dv$

$= \int \frac{\sin a\ dv}{\sqrt{v^2 - \sin^2 a}}$

$= \sin a\ \ell n \left|\sin x + \sqrt{\sin^2 x - \sin^2 a}\right|$

$$= \int \frac{(t^3+1)}{t^3(t^2+1)} 6t^5 dt$$

$$= 6\int \frac{t^2(t^3+1)}{t^2+1} dt$$

$$= 6\int \frac{t^5+t^2}{t^2+1} dt$$

$$= 6\int \frac{(t^2+1)(t^3-t+1)+(t-1)}{t^2+1} dt$$

$$= 6\int \left[(t^3-t+1)+\frac{(t-1)}{t^2+1}\right] dt$$

$$= 6\int \left[\frac{t^4}{4}-\frac{t^2}{2}+t+\frac{1}{2}\int\frac{2t}{t^2+1}dt-\int\frac{1}{t^2+1}dt\right]$$

$$= \boxed{I = 6\left[\frac{t^4}{4}-\frac{t^2}{2}+t+\frac{1}{2}\ell n(t^2+1)-\tan^{-1}t\right]+c}$$

Q.8

Sol. $I = \int \sin^{-1}\sqrt{\frac{x}{a+x}}\ dx$

put $x = a\tan^2\theta$

$dx = a\ 2\tan\theta\ \ \sec^2\theta\ d\theta$

$$= \int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a+a\tan^2\theta}}\ \ a.2\tan\theta\sec^2 d\theta$$

$$= 2a\int \sin^{-1}(\sin\theta)\tan\theta.\sec^2\theta\ d\theta$$

$$= 2a\int \theta.\tan\theta.\sec^2\theta\ d\theta$$

$$= 2a\left[\theta.\int\tan\theta.\sec^2\theta d\theta-\int\left(1.\int\tan\theta\sec^2\theta.d\theta\right)d\theta\right]$$

$$= 2a\ \left[\theta\int t dt-\int(\int t.dt)d\theta\right]$$ put $\tan\theta = t$

$$= 2a\left[\theta.\frac{t^2}{2}-\int\frac{t^2}{2}.d\theta\right]$$ $\sec^2\theta\ d\theta = dt$

put $\ell nx = t = \frac{1}{x}dx = dt$

$$I = \int \tan t.\tan(t - \ell n2)\tan(\ell n2)\ dt$$

$$I = \int (\tan t - \tan(\ell n2) - \tan(t - \ell n2))dt$$

$= \ell n\sec t - t\tan(\ell n2) - \ell n\sec(t - \ell n2) + c$

$= \ell n(\sec(\ell nx)) - \ell n(x).\tan(\ell n2) - \ell n\left(\sec\left(\ell n\frac{x}{2}\right)\right) + c$

Q.11

Sol. $I = \int_1^2 \frac{(x^2-1)dx}{x^3\sqrt{2x^4-2x^2+1}}$

$$= \int_1^2 \frac{(x^2-1)\,dx}{x^5\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}} = \int_1^2 \frac{(x^{-3}-x^{-5})}{\sqrt{2-2x^{-2}+x^{-4}}}\,dx$$

put $2 - 2x^{-2} + x^{-4} = t^2 \Rightarrow (x^{-3} - x^{-5})\,dx = \frac{1}{2}\ tdt$

when $x = 1$ then $t = 1$

&

$x = 2$ then $t = \frac{5}{4}$

$$I = \frac{1}{2}\int_1^{5/4}\frac{t}{t}dt = \frac{1}{2}\left(\frac{5}{4}-1\right) = \frac{1}{8}$$

$I = \frac{u}{v} = \frac{1}{8}$ then $\left(1000\left(\frac{1}{8}\right)\right) = 125$ Ans

Q.12

Sol. Given $\frac{d}{dx}(h(x)) = -\frac{\sin x}{\cos^2(\cos x)}$

$$= \int_0^{\pi/2} \sin^3 x dx + a^3 \int_0^{\pi/2} \cos^3 x dx + 3a \int_0^{\pi/2} \sin^2 x \cos x \, dx + 3a^2 \int_0^{\pi/2} \sin x \cos^2 x \, dx$$

$$= \frac{2}{3} + a^3 . \frac{2}{3} + 3a \int_0^{\pi/2} (1 - \cos^2 x) \cos x \, dx + 3a^2 \int_0^{\pi/2} (1 - \sin^2 x) \sin x \, dx$$

$$= \frac{2}{3} (1 + a^3) + 3a \left(1 - \frac{2}{3}\right) + 3a^2 \left(1 - \frac{2}{3}\right)$$

$$I_1 = \frac{2}{3} + \frac{2a^3}{3} + a + a^2$$

now

$$I_2 = \int_0^{\pi/2} x . \cos x \, dx \Rightarrow x . \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x \, dx$$

$$I_2 = x \sin \Big|_0^{\pi/2} + \cos x \Big|_0^{\pi/2}$$

$$I_2 = \frac{\pi}{2} - 1$$

therefore $I = I_1 - \frac{4a}{\pi - 2} . I_2$

$$2 = \frac{2}{3} + \frac{2a^3}{3} + a + a^2 - \left(\frac{4a^2}{\pi - 2}\right)\left(\frac{\pi - 2}{2}\right)$$

$$2 = \frac{2}{3} + \frac{2a^3}{3} - a + a^2 \Rightarrow 2a^3 + 3a^2 - 3a + 2 = 6$$

$$2a^3 + 3a^2 - 3a - 4 = 0 \quad \Big\langle \; a_1, \; a_2, \; a_3$$

so $\left\{ \begin{array}{c} a_1 + a_2 + a_3 = -\frac{3}{2} \\ a_1 a_2 + a_2 a_3 + a_3 a_1 = -\frac{3}{2} \end{array} \right\} \Rightarrow (a_1 + a_2 + a_3)^2 = a_1^2 + a_2^2 + a_3^2 + 2(a_1 a_2 + a_2 a_3 + a_3 a_2)$

$a_1 a_2 a_3 = 2$ $\qquad \Rightarrow \frac{21}{4}$

Q.15

Sol. $u = \int_0^{\pi/4} \left(\frac{\cos x}{\sin x + \cos x}\right)^2 dx$

$$= 2\int_0^2 \frac{x^2}{\sqrt{x^2+4}}\,dx - 0 = 2\int_0^2 \frac{x^2+4-4}{\sqrt{x^2+4}}\,dx$$

$$= 2\int_0^2 \sqrt{x^2+4}\,dx - 8\int_0^2 \frac{1}{\sqrt{x^2+4}}\,dx$$

$$= 2\int_0^2 \sqrt{x^2+4}\,dx - 8\int_0^2 \frac{1}{\sqrt{x^2+4}}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{x^2+4} + \frac{4}{2}\ell n(x+\sqrt{x^2+4})\right]_0^2 - 8\,\ell n\,(x+\sqrt{x^2+4})\Big|_0^2$$

$$= 4\sqrt{2} - 4\ell n(\sqrt{2}+1) \text{ Ans}$$

$$\sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ell n\left(x+\sqrt{x+a^2}\right)+c$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ell n(x+\sqrt{x^2+a^2})+c$$

Q.18 $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7+3x^6-10x^5-7x^3-12x^2+x+1}{x^2+2}\,dx$

Sol. $\displaystyle I = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7-10x^5-7x^3+x}{x^2+2}\,dx + \int_{-\sqrt{2}}^{\sqrt{2}} \frac{3x^6-12x^2+1}{x^2+2}\,dx$

$$= 0 + 2\int_0^{\sqrt{2}} \frac{3x^6-12x^2+1}{x^2+2}\,dx$$

$$= 2\int_0^{\sqrt{2}} \frac{3x^2(x^4-4)+1}{x^2+2}\,dx = 2\int_0^{\sqrt{2}} \left(3x^2(x^2-2)+\frac{1}{x^2+2}\right)dx$$

$$= 2\int_0^{\sqrt{2}} \left(3x^4-6x^2+\frac{1}{x^2+2}\right)dx$$

$$= 2\left(\frac{3x^5}{5} - 2x^3 + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{2}\right)\right)_0^{\sqrt{2}}$$

Q.20 $\int_0^1 \frac{\sin^{-1}\sqrt{x}}{x^2 - x + 1}dx$

Sol. Put $x = \sin^2\theta \Rightarrow dx = \sin 2\theta\, d\theta$

$$I = \int_0^{\pi/2} \frac{(\theta \sin 2\theta)}{\sin^4\theta - \sin^2\theta + 1}d\theta \qquad ...(1)$$

$$I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - \theta\right)\sin 2\theta}{\cos^4\theta - \cos^2\theta + 1}d\theta$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - \theta\right)\sin 2\theta}{(1-\sin^2\theta)^2 - (1-\sin^2\theta) + 1}d\theta = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - \theta\right)\sin 2\theta}{\sin^4\theta - \sin^2\theta + 1}d\theta \qquad ...(2)$$

(1) + (2) $\pi/2$

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin 2\theta}{\sin^4\theta - \sin^2\theta + 1}d\theta$$

put $\sin^2\theta = t$

$$2I = \frac{\pi}{2}\int_0^1 \frac{dt}{t^2 - t + 1} = \frac{\pi}{2}\int_0^1 \frac{dt}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$2I = \frac{\pi}{2}.\frac{2}{\sqrt{3}}\ \tan^{-1}\left(\frac{\left(t - \frac{1}{2}\right).2}{\sqrt{3}}\right)\Bigg|_0^1$$

$$2I = \frac{\pi}{\sqrt{3}}\left(\frac{2\pi}{6}\right) \Rightarrow I = \frac{\pi^2}{6\sqrt{3}}$$ **Ans**

$I = \frac{\pi}{8}\,\ell n2$ **Ans**

Q.22

Sol. Let $I = \int_{-\frac{1}{n}}^{1/n} (2007\sin x)\,|x|\,dx + \int_{-\frac{1}{n}}^{1/n} (2008\cos x)\,|x|\,dx$

odd vanish

$$I = \int_{-\frac{1}{n}}^{1/n} (2008\cos x)\,|x|\,dx = 2\int_{0}^{1/n} ((2008)\cos x)x\,dx$$

$$= 2.\ 2008 \int_{0}^{1/n} x.\cos x\ dx$$

$$= 2.\ 2008 \left[x.\sin x\Big|_{0}^{1/n} - \int_{0}^{1/n} \sin x\,dx \right]$$

$$= 2.2008\left[\frac{1}{n}\sin\frac{1}{n} + \cos\frac{1}{n} - 1\right]$$

put $n = \frac{1}{y}$

$$= 2.2008\ \lim_{y\to\infty}\left[\frac{y\sin y + \cos y - 1}{y^2}\right]$$

$$= 2.2008\left[1 - \lim_{y\to 0}\frac{1-\cos y}{y^2}\right]$$

$= 2.2008\ .\ \frac{1}{2} = 2008$ **Ans**

$$= \int_0^{\pi} \sqrt{1+2(1+\cos 2x)+4\cos x}\ dx$$

$$= \int_0^{\pi} \sqrt{1+2.2\cos^2 x+4\cos x}\ dx$$

$$= \int_0^{\pi} \sqrt{4\cos^2 x+4\cos x+1}\ dx$$

$$= \int_0^{\pi} |2\cos x+1|\ dx = 2\sqrt{3}+\frac{5\pi}{3}$$

$$= \frac{\pi}{3/5}+\sqrt{12}$$

compare with $\left(\frac{\pi}{k}+\sqrt{w}\right)$ then

$k = \frac{3}{5}$; $w = 12$

so $k^2 + w^2 = \frac{9}{25}+144$

$$= \frac{3609}{25}$$ **Ans**

Q.25

Sol. $$= \int_0^1 \frac{(1-x^2)}{(1+x^2+2x)\sqrt{x+x^2+x^3}}dx$$

$$= \int_0^1 \frac{\left(1-\frac{1}{x^2}\right)}{\left(\frac{1}{x}+x+2\right)\sqrt{x+\frac{1}{x}+1}}dx$$

put $x + \frac{1}{x} + 1 = t^2$

$$I = \frac{a+b}{\sqrt{2}} \int_0^{\pi/2} dx \Rightarrow I = \frac{a+b}{2\sqrt{2}}\pi \textbf{ Ans}$$

Q.27

Sol. put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

when $x = 0 \Rightarrow \theta = 0$

& $x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$I = \int_0^{\pi/2} \frac{\sin^2\theta\, \ell n(\sin\theta)}{\cos\theta}.\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \left(\frac{1-\cos 2\theta}{2}\right).\ell n \sin\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \ell n \sin\theta\, d\theta - \frac{1}{2}\int_0^{\pi/2} \cos 2\theta\ \ell n(\sin\theta) d\theta$$

$$= \frac{1}{2}\left(-\frac{\pi}{2}\ell n^2\right) - \frac{1}{2}\left[\ell n \sin\theta.\frac{\sin 2\theta}{2}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta}.\frac{\sin 2\theta}{2} d\theta$$

$$= -\frac{\pi}{4}\ell n 2 + \frac{1}{2}\int_0^{\pi/2} \cos^2\theta\, d\theta = \frac{\pi}{8}(1 - \ell n 4) \textbf{ Ans}$$

Q.28

Sol. $$I = \int_{\pi/4}^{\pi/3} \frac{(\sin^3\theta - \cos^3\theta - \cos^2\theta)}{\sin^2\theta\cos^2\theta}\left(\frac{\sin\theta + \cos\theta + \cos^2\theta}{\sin\theta\cos\theta}\right)^{2007} d\theta$$

$$= \int_{\pi/4}^{\pi/3} (\tan\theta\sec\theta - \cot\theta\,\mathrm{cosec}\,\theta - \mathrm{cosec}^2\theta)\ (\sec\theta + \mathrm{cose}\theta + \cot\theta)^{2007} d\theta$$

put $\sec\theta + \mathrm{cosec}\theta + \cot\theta + t$

$(\sec\theta \tan\theta - \mathrm{cosec}\theta \cot\theta - \mathrm{cosec}^2\theta)\, d\theta = dt$

when $\theta = \pi/4$

$$= (\pi+3).2\int_0^{2/\pi} \frac{\sin x}{1+\cos^2 x}dx$$

put cos x = t $\Rightarrow$ – sinx dx = dt

where $x = 0 \Rightarrow t = 1$

&

$$x = \frac{\pi}{2} \Rightarrow t = 0$$

$$= (\pi + 3) . 2\int_0^1 \frac{dt}{1+t^2}$$

$$= (\pi+3)\tan^{-1} t\Big|_0^1 = (\pi + 3)\ \frac{\pi}{2}$$

Q.30

Sol. $$I = \int_0^{\pi} \frac{(ax+b)\ \sec x \tan x}{\sec^2+3}\ dx \qquad ...(1)$$

use prop $\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx$

$$I = \int_0^{\pi} \frac{(a\pi - ax + b)\sec x \tan x}{\sec^2 x + 3}\ dx \qquad ...(2)$$

(1) + (2)

$$2I = \int_0^{\pi} \frac{(a\pi + 2b)\sec x \tan x}{\sec^2 x + 3}\ dx$$

use prop $\int_0^{2\pi} f(x)dx = 2\int_0^{\pi} f(x)dx$

$$2I = 2(a\pi + 2b)\int_0^{\pi/2} \frac{\sec x \tan x}{\sec^2+3}dx$$

put sec x = t $\Rightarrow$ sec x tan x dx = dt

when $x = 0 \Rightarrow t = 1$ &