Exercise 2(A)

1 [Hint: $I = \int_1^{\infty} \frac{dx}{(e \cdot e^x + e^3 \cdot e^{-x})} = \int_1^{\infty} \frac{e^x\,dx}{e(e^{2x}+e^2)}$ (multiply N^r and D^r by e^x)

put $e^x = t \quad \Rightarrow \quad e^x\,dx = dt$

$I = \frac{1}{e}\int_e^{\infty} \frac{dt}{t^2+e^2} = \frac{1}{e^2}\tan^{-1}\frac{t}{e}\Big|_e^{\infty} = \frac{1}{e^2}\left[\frac{\pi}{2}-\frac{\pi}{4}\right] = \frac{\pi}{4e^2}$ Ans.]

2 [Hint: put $e^{x^2} = t$; $e^{x^2} \cdot 2x\,dx = dt$; $\int_1^{\pi/2} \cos t\,dt = [\sin t]_1^{\pi/2} = 1 - (\sin 1)$]

3 [Hint: Note that in $\left(-\frac{1}{2},\frac{1}{2}\right)$, $\sin^{-1}(3x-4x^3) = 3\sin^{-1}x$ and $\cos^{-1}(4x^3-3x) = 2\pi - 3\cos^{-1}x$

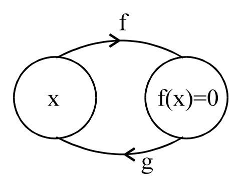
hence $f(x) = 3\sin^{-1}x - 2\pi + 3\cos^{-1}x = -\frac{\pi}{2}$

$\therefore I = -\frac{\pi}{2}\int_{-1/2}^{1/2} dx = -\frac{\pi}{2}$]

[Alternate: $f(x) = \sin^{-1}(3x-4x^3) - [\pi - \cos^{-1}(3x-4x^3)]$

$= -\pi + (\sin^{-1}(3x-4x^3) + \cos^{-1}(3x-4x^3)) = -\frac{\pi}{2}$]

4 [Sol. $f'(x) = \frac{1}{\sqrt{1+x^4}} = \frac{dy}{dx}$



now $g'(x) = \frac{dx}{dy} = \sqrt{1+x^4}$

when $y = 0$ i.e. $\int_2^x \frac{dt}{\sqrt{1+t^4}} = 0$ then $x = 2$ (think !)

hence $g'(0) = \sqrt{1+16} = \sqrt{17}$]

5 [Sol. $l = \ln \lim_{t\to 0} \frac{\int_0^t (1+a\sin bx)^{c/x}\,dx}{t} = \ln \lim_{t\to 0} (1+a\sin bt)^{c/t}$ (using L'Hospital's rule)

$$= \ln e^{\lim_{t\to 0}\frac{c}{t}(a\sin bt)} = \lim_{t\to 0}\frac{abc\sin bt}{bt} = abc \text{ Ans.]}$$

6 [Sol. $\sin nx - \sin(n-2)x = 2\cos(n-1)x\sin x$

$$\int \frac{\sin nx}{\sin x}dx = \int 2\cos(n-1)\,dx + \int \frac{\sin(n-2)x}{\sin x}dx$$

$$\therefore \int_0^{\pi/2}\frac{\sin 5x}{\sin x}dx = \int_0^{\pi/2} 2\cos 4x\,dx + \int_0^{\pi/2}\frac{\sin 3x}{\sin x}dx = 0 + \int_0^{\pi/2}\frac{\sin 3x}{\sin x}dx = \int_0^{\pi/2}dx = \frac{\pi}{2} \text{ Ans.]}$$

7 [Sol. $F(x) = \frac{1}{2}\int \frac{(x^2+1)-(x-1)^2}{(x^2+1)(x-1)}dx = \frac{1}{2}\ln|x-1| - \frac{1}{2}\int \frac{x-1}{x^2+1}dx$

$$= \frac{1}{2}\ln|x-1| + \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x + C$$

$\therefore$ discontinuous at $x = 1$

note that $f(x) = \int \frac{dx}{x^{1/3}} = \frac{3}{2}x^{2/3} + C$ is continuous although $\frac{1}{x^{1/3}}$ is discontinuous at $x = 0$]

8 [Sol. $T_r = \dfrac{1}{\sqrt{\frac{r}{n}}\cdot n\left(3\sqrt{\frac{r}{n}}+4\right)^2}$

$$S = \frac{1}{n}\sum_{1}^{4n}\frac{1}{\left(3\sqrt{\frac{r}{n}}+4\right)^2\cdot\sqrt{\frac{r}{n}}} = \int_0^4 \frac{dx}{\sqrt{x}(3\sqrt{x}+4)^2}$$

put $3\sqrt{x}+4 = t \Rightarrow \frac{3}{2}\frac{1}{\sqrt{x}}dx = dt$

$$= \frac{2}{3}\int_4^{10}\frac{dt}{t^2} = \frac{2}{3}\left[\frac{1}{t}\right]_{10}^{4} = \frac{2}{3}\left[\frac{1}{4}-\frac{1}{10}\right] = \frac{2}{3}\cdot\frac{6}{40} = \frac{1}{10} \quad]$$

9 [Sol. $f'(x) = f(x) \Rightarrow f(x) = Ce^x$ and since $f(0) = 1$

$\therefore$ $1 = f(0) = C$ $\therefore$ $f(x) = e^x$ and hence $g(x) = x^2 - e^x$

Thus $\int_0^1 f(x)g(x)\,dx = \int_0^1 (x^2e^x - e^{2x})\,dx$

$$= x^2e^x\Big|_0^1 - 2\int_0^1 xe^x\,dx - \left[\frac{e^{2x}}{2}\right]_0^1 = (e-0) - 2\left[xe^x\Big|_0^1 - e^x\Big|_0^1\right] - \frac{1}{2}(e^2-1)$$

$= (e-0) - 2[(e-0)-(e-1)] - \frac{1}{2}(e^2-1)$

$= e - \frac{1}{2}e^2 - \frac{3}{2}$]

10 [Sol. $I = \int_{-\pi/2}^{\pi/2} \frac{\cos\theta \, d\theta}{(2-\sin\theta)\cos\theta}$ (putting $x = \sin\theta$)

$= \int_0^{\pi/2} \left(\frac{1}{2-\sin\theta} + \frac{1}{2+\sin\theta} \right) d\theta$ $\left[\text{using } \int_{-a}^{a} f(x)\,dx = \int_0^a [f(x)+f(-x)\,dx \right]$

$= 4\int_0^{\pi/2} \frac{d\theta}{4-\sin^2\theta} = \frac{4}{3}\int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{\frac{4}{3}+\tan^2\theta} = \frac{4}{3}\int_0^{\infty} \frac{d\theta}{t^2+\frac{4}{3}} = \frac{4}{\sqrt{3}}.\frac{\sqrt{3}}{2}.\tan^{-1}\frac{\sqrt{3}\,t}{2}\Bigg|_0^{\infty} = \frac{2}{\sqrt{3}}.\frac{\pi}{2} = \frac{\pi}{\sqrt{3}}$]

11 [Sol. $T_r = \frac{\pi}{6n}\sec^2\frac{r\pi}{6n}$

$S = \sum T_r = \frac{\pi}{6n}\sum_{r=1}^{n}\sec^2\frac{r\pi}{6n} = \frac{\pi}{6}\int_0^1 \sec^2\frac{\pi x}{6}\,dx = \tan\frac{\pi x}{6}\Bigg|_0^1 = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$]

12 [Sol. Clearly f is an even function, hence

$I_1 = \int_0^{\pi} f(\cos(\pi - x))dx = \int_0^{\pi} f(-\cos x)\,dx = \int_0^{\pi} f(\cos x)\,dx$

$\therefore$ $I_1 = 2\int_0^{\pi/2} f(\cos x)\,dx = 2I_2$ $\Rightarrow$ $\frac{I_1}{I_2} = 2$ **Ans.**

<u>Alternatively</u>: let $u = \cos x$ $\Rightarrow$ $du = -\sin x\,dx$

$\therefore$ $I_1 = \int_{-1}^{1} \frac{f(u)}{\sqrt{1-u^2}}\,du$ $\Rightarrow$ $2\int_0^1 \frac{f(u)}{\sqrt{1-u^2}}\,du$ ....(1)

|||ly with $\sin t = t$, $I_2 = \int_0^1 \frac{f(t)}{\sqrt{1-t^2}}\,dt$ ....(2)

from (1) and (2) $\frac{I_1}{I_2} = 2$ Ans.]

13 [Hint: $\int_2^4 \left(\frac{\ln 2}{\ln x} - \frac{\ln 2}{\ln^2 x} \right) dx$ if $f(x) = \frac{1}{\ln x}$ $\Rightarrow$ $x f'(x) = -\frac{1}{\ln^2 x}$

$\Rightarrow \quad I = \ln 2 \left(\frac{x}{\ln x}\right)_2^4 = \ln 2\left[\frac{4}{\ln 4} - \frac{2}{\ln 2}\right] = 0$]

14 [Hint: On rationalisation,

$$\int_{-1}^{1}\frac{(1+x^3)-\sqrt{1+x^6}}{1+x^6+2x^3-1-x^6}dx = \int_{-1}^{1}\frac{(1+x^3)-\sqrt{1+x^6}}{2x^3}dx = \underbrace{\frac{1}{2}\int_{-1}^{1}\frac{1}{x^3}dx}_{\text{odd} \Rightarrow \text{zero}} + \frac{1}{2}\int_{-1}^{1}dx - \underbrace{\int_{-1}^{1}\frac{\sqrt{1+x^6}}{2x^3}dx}_{\text{odd} \Rightarrow \text{zero}}$$

$\frac{1}{2}\int_{-1}^{1}dx = \frac{1}{2}\cdot 2 = 1$ Ans.]

15 [Sol. at y = 0, x = 2

$f'(x) = \sqrt{9+x^4}\cdot 2x$

$\therefore \quad g'(y) = \left.\frac{1}{f'(x)}\right|_{x=2} = \frac{1}{2x\sqrt{9+x^4}} = \frac{1}{20}$]

16 [Sol. $\left.\frac{t^3}{3}\right|_0^{f(x)} = x\cos\pi x \quad \Rightarrow [f(x)]^3 = 3x\cos\pi x$ ....(1)

$[f(9)]^3 = -27 \Rightarrow f(9) = -3$

also differentiating $\int_0^{f(x)} t^2\,dt = x\cos\pi x$

$[f(x)]^2\cdot f'(x) = \cos\pi x - x\pi\sin\pi x$

$\therefore \quad [f(9)]^2\cdot f'(9) = -1$

$\Rightarrow \quad f'(9) = -\frac{1}{(f(9))^2} = -\frac{1}{9} \qquad f'(9) = -\frac{1}{9} \quad \Rightarrow$ (A)]

17 [Hint: $\lim_{x\to\infty}\frac{x^{3/2}}{(x-1)}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{2[1-(1/x)]} = \frac{1}{2}$ Ans.]

18 [Sol. $I = \int_1^e \underbrace{f''(x)}_{II}\,\underbrace{\ln x}_{I}\,dx = \ln x\cdot f'(x)\Big|_1^e - \int_1^e \frac{f'(x)}{x}dx$

$I = 1 - I_1$

$I_1 = \int_1^e \frac{1}{x}f'(x)\,dx = \left.\frac{1}{x}\cdot f(x)\right|_1^e + \int_1^e\frac{f(x)}{x^2}dx$

$= \left(\frac{1}{e}-1\right) + \frac{1}{2}$

$= \frac{1}{e} - \frac{1}{2}$

$\therefore \quad I = 1 - \frac{1}{e} + \frac{1}{2} = \frac{3}{2} - \frac{1}{e}$ Ans.]

19 [Sol. $f'(x)\frac{dy}{dx} = \frac{1}{\sqrt{x^4+3x^2+13}}$ when $y = f(x)$

$\therefore \quad g'(y) = \frac{1}{dy/dx} = \sqrt{x^4+3x^2+13}$

when y = 0 then x = 3

hence $g'(0) = \sqrt{3^4+27+13} = \sqrt{121} = 11$ Ans.]

20 [Hint: $I = \int \sqrt{1+2\,\text{cosec}\,x\,\cot x + 2\cot^2 x}$

$= \int \sqrt{\text{cosec}^2 x + 2\,\text{cosec}\,x\cot x + \cot^2 x}\,dx$

$= \int (\text{cosec}\,x + \cot x)\,dx$]

21 [Hint: $\left.\frac{t^2}{2} - \log_2 a \cdot t\right|_0^2 = 2 - \log_2(a^2)$

$(2 - 2\log_2 a) = 2 - 2\log_2 a$

$2\log_2 a = 2\log_2 a \quad \Rightarrow \quad a \in R^+$]

22 [Hint: Put $4x - 5 = 5t^2 \Rightarrow 4dx = 10t\,dt$ or <u>better will be $5(4x-5) = t^2$</u>]

$$I = \frac{5}{2}\int_{\sqrt{\frac{3}{5}}}^{\sqrt{\frac{7}{5}}} \left(\sqrt{\frac{5}{2}(1+t^2)-5t} + \sqrt{\frac{5}{2}(1+t^2)+5t}\right) t\,dt = \left(\frac{5}{2}\right)^{3/2}\int_{\sqrt{\frac{3}{5}}}^{\sqrt{\frac{7}{5}}}\left(|t-1| + |(t+1)|\right)t\,dt$$

$$= \left(\frac{5}{2}\right)^{3/2}\left[\int_{\sqrt{\frac{3}{5}}}^{1}\left((1-t) + |(1+t)|\right)t\,dt + \int_{1}^{\sqrt{\frac{7}{5}}}\left((t-1) + (t+1)\right)t\,dt\right]$$

$$= \left(\frac{5}{2}\right)^{3/2}\left[2\int_{\sqrt{\frac{3}{5}}}^{1} t\,dt + \int_{1}^{\sqrt{\frac{7}{5}}} t^2\,dt\right]$$]

23 [Hint: $\frac{dy}{dx} = \frac{1}{\sqrt{y^2+1}}$

$$\frac{dy}{dx} = \sqrt{y^2+1}\,; \quad \frac{d^2y}{dx^2} = \frac{y}{\sqrt{y^2+1}}\sqrt{y^2+1} = y \text{ Ans.]}$$

24 [Hint: $f(x) = \sqrt{1+x^2} - x\,;\ \lim_{x\to-\infty} x\left(\sqrt{1+x^2} - x\right) \to -\infty \Rightarrow$ DNE]

25 [Sol. $x = \frac{1}{t} \Rightarrow dx = \frac{1}{t^2}\,dt$

$$I = \int_2^{1/2} t\,\sin\left(\frac{1}{t} - t\right)\left(-\frac{1}{t^2}\right) dt = \int_2^{1/2} \frac{1}{t}\sin\left(t - \frac{1}{t}\right) dt = -\int_{1/2}^{2} \frac{1}{t}\sin\left(t - \frac{1}{t}\right) dt = -I$$

$\Rightarrow$ $2I = 0 \Rightarrow I = 0$

Alternatively : put $x = e^t \Rightarrow I = \int_{-\ln 2}^{\ln 2} \sin(e^t - e^{-t})dt = 0$ (odd function)]

26 [Sol. $f'(\ln x) = \begin{cases} 1 & \text{for } 0 < x \le 1 \\ x & \text{for } x > 1 \end{cases}$

put $\ln x = t \Rightarrow x = e^t$

for $x > 1\,;\ f'(t) = e^t$ for $t > 0$

integrating $f(t) = e^t + C\ ;\ f(0) = e^0 + C \Rightarrow C = -1$

$\therefore$ $f(t) = e^t - 1$ for $t > 0$ (corresponding to $x > 1$)

$\therefore$ $f(x) = e^x - 1$ for $x > 0$ ....(1)

again for $0 < x \le 1$

$f'(\ln x) = 1$ $(x = e^t)$

$f'(t) = 1$ for $t \le 0$

$f(t) = t + C$

$f(0) = 0 + C \Rightarrow C = 0 \Rightarrow f(t) = t$ for $t \le 0 \Rightarrow f(x) = x$ for $x \le 0$]

27 [Sol. $\int \frac{1}{x}\ln\frac{x}{e^x}dx = \int \frac{1}{x}(\ln x - \ln e^x)\,dx$

$$= \int \frac{\ln x - x}{x}dx = \left[\int \frac{1}{x}\ln x\,dx - \int \frac{1}{x}dx\right] \quad (\text{put } \ln x = u\,;\ \frac{1}{x}dx = du\ )$$

$$= \int u\,dx - \int 1\,dx = \frac{1}{2}\ln^2 x - x + C$$]

28 [Sol. $\int_1^e e^x[x\ln x + 1 + \ln x - 1]dx = \int_1^e e^x[(\underbrace{x\ln x}_{f(x)}) + (\underbrace{\ln x + 1}_{f'(x)})]dx - \int_1^e e^x dx$

$= e^x\cdot(x\ln x)\Big|_1^e - \left[e^x\right]_1^e = (e^e\cdot e - 0) - [e^e - e]$

$= e^e(e - 1) + e$ Ans.]

29 [Hint: $\left|\int_{10}^{19} \frac{\sin x}{1+x^8}\right| \le \int_{10}^{19} \frac{|\sin x|}{1+x^8}\,dx \le \int_{10}^{19} \frac{dx}{1+x^8} < \int_{10}^{19} \frac{dx}{x^8} = \left[\frac{x^{-7}}{-7}\right]_{10}^{19}$

$= -\frac{1}{7}\,[19^{-7} - 10^{-7}] = \frac{1}{7}\,[10^{-7} - 19^{-7}] < 10^{-7}$]

30 [Sol. $\lim_{n\to\infty} \int_0^2 \left(1+\frac{t}{n+1}\right)^n dt = \lim_{n\to\infty}\left[\left(1+\frac{t}{n+1}\right)^{n+1}\right]_0^2 = \lim_{n\to\infty}\left(1+\frac{2}{n+1}\right)^{n+1} - 1 = e^2 - 1$

note that $\left[\left(1+\frac{t}{n+1}\right) \textit{is a linear function } a+bt \textit{ type}\right]$]

31 [Sol. $I = \int x\, 2^{\ln(x^2+1)} dx$ let $x^2 + 1 = t$; $x\,dx = \frac{dt}{2}$

Hence $I = \frac{1}{2}\int 2^{\ln t} dt = \frac{1}{2}\int t^{\ln 2} dt = \frac{1}{2}\cdot\frac{t^{\ln 2+1}}{\ln 2+1} + C = \frac{1}{2}\cdot\frac{(x^2+1)^{\ln 2+1}}{\ln 2+1} + C \Rightarrow$ (C)]

32 [Hint: $\int_0^1 (1+\cos^8 x)\, f(x)\, dx = \int_0^2 (1+\cos^8 x)\, f(x)\, dx =$

$\int_0^1 (1+\cos^8 x)\, f(x)\, dx + \int_1^2 (1+\cos^8 x)\, f(x)\, dx$

Hence $\int_1^2 (1+\cos^8 x)\, f(x)\, dx = 0$

$\Rightarrow (1+\cos^8 x)\, f(x) = 0$ at least once in (1,2)

but $1 + \cos^8 x \ne 0$

$\Rightarrow f(x) = ax^2 + bx + c$ vanishes at least once in (1,2)]

33 [Hint: $I = \int_0^{\pi/4} (1 - 2\sin^2 x)^{3/2} \cos x\, dx$. Put $\sqrt{2}\,\sin x = \sin\theta$

$\Rightarrow I = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{3\pi}{16\sqrt{2}}$]

34 [Sol. Given $\int f(x)\,dx = g(x) \Rightarrow g'(x) = f(x)$

now $\frac{d}{dx}\left(\ln(1+g^2(x))\right) = \frac{2g(x)g'(x)}{1+g^2(x)} = \frac{2f(x)g(x)}{1+g^2(x)} \Rightarrow$ (B)]

35 [Sol. $\lim_{x\to 0} \dfrac{\int_0^x \sin t^2 dt}{x^3 \dfrac{(1-\cos x)}{x^2}}$ (using $\lim_{x\to 0} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$)

$= \lim_{x\to 0} \dfrac{\int_0^x \sin t^2 dt}{x^3}$ (Using L'Hospital Rule)

$2 \lim_{x\to 0} \dfrac{\sin x^2}{3x^2} = \dfrac{2}{3}$ Ans.]

36 [Sol. $I = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} f(-x)\,dx$ (using K)

$2I = \int_{-1}^{1} \left(f(x) + f(-x)\right) dx = \int_{-1}^{1} (x^2)\,dx$

$2I = 2\int_{0}^{1} (x^2)\,dx \quad \Rightarrow \quad I = \int_{0}^{1} (x^2)\,dx = \dfrac{1}{3}$ Ans.]

37 [Sol. $I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \dfrac{x^4}{1-x^4} \cos^{-1} \dfrac{2x}{1+x^2}\,dx$ ....(1)

$I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \dfrac{x^4}{1-x^4} \cos^{-1}\left(\dfrac{-2x}{1-x^4}\right) dx$ (using King)

$I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \dfrac{x^4}{1-x^4} \left(\pi - \cos^{-1} \dfrac{2x}{1-x^4}\right) dx$ ....(2)

add (1) and (2)

$\therefore \quad 2I = \pi \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \dfrac{x^4}{1-x^4}\,dx$

$2I = 2\pi \int_{0}^{1/\sqrt{3}} \dfrac{x^4}{1-x^4}\,dx$

$\therefore \quad k = \pi$ Ans.]

38 [Sol. $I = \int_0^{\pi/2} \sqrt{\tan x}\, dx$ ....(1) ; $I = \int_0^{\pi/2} \sqrt{\cot x}\, dx$ ....(2)

adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx = \sqrt{2}\int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx = \sqrt{2}\int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}} dx$$

$$= \sqrt{2}\int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} = 2\sqrt{2}\int_0^{1} \frac{dt}{\sqrt{1-t^2}} = \sqrt{2}\,\pi \quad (\text{where } \sin x - \cos x = t)$$

$\therefore \quad I = \frac{\pi}{\sqrt{2}}$ Ans.]

39 [Hint: $I_1 = \int_{-\pi/4}^{\pi/4} \ln(\sin x + \cos x)\, dx = \int_{-\pi/4}^{\pi/4} \ln(\cos x - \sin x)\, dx$ (using king)

$$\Rightarrow 2 I_1 = \int_{-\pi/4}^{\pi/4} \ln \cos 2x\, dx = 2\int_0^{\pi/4} \ln(\cos 2x) = \int_0^{\pi/2} \ln(\cos t)\, dt \text{ where } 2x = t$$

$$\int_0^{\pi/2} \ln(\sin t)\, dt = I \Rightarrow I_1 = I/2 \text{]}$$

40 [Hint: $f'(x) = \frac{1}{x} + \pi\cos(\pi x) + C$

$f'(2) = \frac{1}{2} + \pi + C = \frac{1}{2} + \pi \quad \Rightarrow \quad C = 0$

$f(x) = \ln|x| + \sin(\pi x) + C'$

$f(1) = C' = 0$

$f(x) = \ln|x| + \sin(\pi x)$]

41 [Hint: $f'(x) = 1 + \ln^2 x + 2\ln x = 0 \Rightarrow (1 + \ln x)^2 = 0 \Rightarrow x = \frac{1}{e}$

Hence $f\left(\frac{1}{e}\right) = 1 + \frac{1}{e} + \int_1^{1/e} (\ln^2 t + 2\ln t)\, dt = 1 + \frac{1}{e} + t\ln^2 t\Big]_1^{1/e} = 1 + \frac{1}{e} + \frac{1}{e} = 1 + 2e^{-1} \Rightarrow$ [D]

42 [Sol. $I = \int_{-\infty}^{\infty} \underbrace{h'(x)}_{II} \cdot \underbrace{\sin x}_{I}\, dx = \sin x \cdot h(x)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \cos x \cdot h(x)\, dx = 0 - \cos 0 = -1 \Rightarrow$ (A)

note that here $\cos x = f(x)$]

43 [Sol. $I = \int_0^{\infty} (x^2)^n . x\, e^{-x^2}\, dx$ put $x^2 = t \Rightarrow x\, dx = -dt/2$

$$= \frac{1}{2}\int_0^{\infty} t^n\, e^{-t}\, dt \;=\frac{1}{2}\left[\; t^n e^{-t}\Big]_0^{\infty} + n\int_0^{\infty} t^{n-1}\, e^{-t}\, dt\;\right] = \frac{1}{2}\left[\; 0 + n\int_0^{\infty} t^{n-1}\, e^{-t}\, dt\;\right]$$

Hence $I = \frac{n!}{2}$]

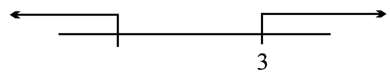
44 [Sol. $\int_a^0 3^{-x}\,(3^{-x} - 2)\,dx \ge 0$ put $3^{-x} = t \Rightarrow 3^{-x}\ln 3\,dx = -dt$

$$\ln 3\int_1^{3^{-a}} (t-2)\,dt \ge 0 \quad \Rightarrow \quad \frac{t^2}{2} - 2t\Bigg]_1^{3^{-a}} \ge 0$$

$$\left(\frac{3^{-2a}}{2} - 2.3^{-a}\right) - \left(\frac{1}{2} - 2\right) \ge 0$$

$3^{-2a} - 4\times 3^{-a} + 3 > 0$

$(3^{-a} - 3)\,(3^{-a} - 1) > 0$



$3^{-a} > 3^1 \Rightarrow a < 1$

or $3^{-a} < 3^0 \Rightarrow a > 0$

Hence $a \in (-\infty, -1) \cup [0, \infty)$]

45 [Sol. $\sin(x + \alpha^2)\Big]_0^{\alpha} = \sin\alpha$

$\sin(\alpha^2 + \alpha) - \sin\alpha^2 = \sin\alpha$

$2\cos(\alpha^2 + \alpha/2)\,\sin\alpha/2 = \sin\alpha$

now proceed and get

$\sqrt{2\pi}\,, \frac{-1 + \sqrt{1 + 8\pi}}{2} \Rightarrow$ 2 solutions]

46 Let $A = \int_0^1 \frac{e^t\,dt}{1+t}$ then $\int_{a-1}^{a} \frac{e^{-t}\,dt}{t-a-1}$ has the value

(A) Ae^{-a} (B*) $-Ae^{-a}$ (C) $-ae^{-a}$ (D) Ae^{a}

[Hint : $I = \int_{a-1}^{a} \frac{e^{-t}}{t-a-1}\,dt$ put $t = a-1+y$ (so that lower limit becomes zero)

$\therefore I = \int_0^1 \frac{e^{1-a-y}}{y-2}\,dy$ (now using king)

$$I = \int_0^1 \frac{e^{1-a-1+y}}{1-y-2}\,dy = -e^{-a}\int_0^1 \frac{e^y}{1+y}\,dy = -e^{-a}A \Rightarrow \text{(B)} \;]$$

47 [Hint: $I = \int_0^1 \frac{e^t\,(t+1-t)}{(1+t)^2}\,dt = \int_0^1 \frac{e^t}{1+t}\,dt - \int_0^1 e^t\left(\frac{1}{1+t} - \frac{1}{(1+t)^2}\right)dt$

$$= A - \left.\frac{e^t}{1+t}\right]_0^1 = A - \frac{e}{2} + 1 \;;\; \text{Alternatively I. B. P. directly]}$$

48 [Hint: $\beta + \int_0^1 \underbrace{x}_{I}\underbrace{2xe^{-x^2}}_{II} dx = \int_0^1 e^{-x^2}\, dx$

$$\beta + \left[-xe^{-x^2}\right]_0^1 - \int_0^1 -e^{-x^2}\, dx = \int_0^1 e^{-x^2}\, dx \qquad \beta = \frac{1}{e} \;]$$

49 [Sol. $g(x) = \int_0^x t \sin\frac{1}{t}\, dt$

$g'(x) = x \sin(1/x)$ which is diff $\Rightarrow$ g is cont. in $(0, \pi)$

$$l(x) = \begin{cases} x \sin x & 0 < x < \pi/2 \\ -\dfrac{\pi \sin x}{2} & \pi/2 < x < \pi \end{cases}$$

obvious discontinuity at $x = \pi/2 \Rightarrow$ (D)]

50 [Sol. $f(x) = \int_0^\pi \frac{t \sin t}{\sqrt{1+\tan^2 x \sin^2 t}} dt$

Using king and add.

$$f(x) = \frac{\pi}{2}\int_0^\pi \frac{\sin t}{\sqrt{1+\tan^2 x \sin^2 t}} dt \qquad = \pi \int_0^{\pi/2} \frac{\sin t}{\sqrt{1+\tan^2 x(1-\cos^2 t)}} dt$$

$$= \pi \int_0^{\pi/2} \frac{\sin t}{\sqrt{\sec^2 x - \tan^2 x \cos^2 t}} dt \;=\; \pi\int_0^1 \frac{dy}{\sqrt{\sec^2 x - \tan^2 x . y^2}}$$

$$= \frac{\pi}{\tan x}\int_0^1 \frac{dy}{\sqrt{\operatorname{cosec}^2 x - y^2}} \quad = \frac{\pi}{\tan x}\left\{\sin^{-1}\frac{y}{\operatorname{cosec} x}\right\}_0^1 = \frac{\pi}{\tan x}\sin^{-1}(\sin x) = \frac{\pi x}{\tan x} \quad]$$

51 [Sol. $I = \int_0^{n\pi+V} |\cos x|\, dx = \underbrace{\int_0^{n\pi} |\cos x|\, dx}_{2n} + \underbrace{\int_{n\pi}^{n\pi+V} |\cos x|\, dx}_{I_1 \text{ (put } x=n\pi+t)}$

$$\text{So, } I_1 = \int_0^V |\cos t|\, dt = \int_0^{\pi/2} \cos t\, dt - \int_{\pi/2}^V \cos x\, dx$$

$= 1 - (\sin x)_{\pi/2}^{V} = 1 - \sin V + 1$

$\therefore \quad I = 2n + 2 - \sin V$]

52 [Sol. $\int \frac{p\,x^{p+2q-1} - qx^{q-1}}{(x^{p+q}+1)^2}\,dx = \int \frac{p\,x^{p-1} - qx^{-q-1}}{(x^p + x^{-q})^2}\,dx$

taking x^q as x^{2q} common from Denominator and take it in N^r]

53 [Hint: for $0 < x < \ln 2$, $[2e^{-x}] = 1$, otherwise zero $\Rightarrow I = \int_0^{\ell n 2} dx + \int_{\ell n 2}^{\infty} 0\,dx = \ln 2$

Alternatively: Put $e^{-x} = t$; $-x = \ln t$; $dx = -\frac{1}{t}dt$; Hence $I = -\int_1^0 \frac{[2t]dt}{t} = \int_0^1 \frac{[2t]dt}{t}$

$I = \int_0^{1/2} 0\,dt + \int_{1/2}^{1} \frac{dt}{t} = \ln t\Big]_{1/2}^{1} = 0 - \ln\frac{1}{2} = \ln 2$ Ans.]

54 [Sol. $2\int_0^1 \frac{dx}{\sqrt{x}} = \left[\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_0^1 = 4\left[\sqrt{x}\right]_0^1 = 4 \Rightarrow$ (C)]

55 [Sol. $I = \int_0^1 x\,\ln\left(\frac{x+2}{2}\right)dx = \int_0^1 x\,(\ln(x+2) - \ln 2)\;dx$

$\therefore \quad I = \int_0^1 x\,\ln(x+2)dx - \ln 2\int_0^1 x\;dx$; hence $I = \ln(x+2).\frac{x^2}{2}\Big]_0^1 - \int_0^1 \frac{x^2}{x+2}\,dx - \frac{\ln 2}{2}$

$= \frac{1}{2}\ln 3 - \int_0^1 \frac{x^2 - 4 + 4}{x+2}\,dx - \frac{\ln 2}{2} \Rightarrow \frac{1}{2}\ln\frac{3}{2} - \int_0^1 \left((x-2) + \frac{4}{x+2}\right)dx$ now proceed]

56 [Sol. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\left(x + \sqrt{x}\right)dx$; put $x = t^2$; $dx = 2t\,dt$

$= \int e^t(t^2 + t)\;dt = e^t(At^2 + Bt + C)$ (Let)

Diffrentiate both the sides

$e^t(t^2 + t) = e^t(2At + B) + (At^2 + Bt + C)\,e^t$

On comparing coefficient we get

$A = 1$; $B = -1$; $C = 1$

57 [Hint: $I = \int_{-1}^{1} \frac{x^3}{x^2+2|x|+1}\, dx + \int_{-1}^{1} \frac{|x|+1}{(|x|+1)^2}\, dx \Rightarrow 2\int_0^1 \frac{dx}{1+x} = 2 \ln 2$]

odd $\Rightarrow$ vanishes even]

58 [Hint: Let $I = \int_0^{\pi/2} \frac{\sin x\, dx}{1+\sin x+\cos x}$

$I = \int_0^{\pi/2} \frac{\cos x}{1+\sin x+\cos x} \Rightarrow 2I = \int_0^{\pi/2} \frac{\sin x+\cos x+1-1}{\sin x+\cos x+1}\, dx$

$\Rightarrow 2I = \frac{\pi}{2} - \ln 2 \Rightarrow I = \frac{\pi}{4} - \frac{1}{2}\ln 2$]

59 [Sol. $\underset{x\to x_1}{\text{Limit}} \frac{\int_{x_1}^{x} f(t)\, dt}{\left(\frac{x-x_1}{x}\right)} = \underset{x\to x_1}{\text{Limit}} \frac{f(x).x^2}{x_1}$ (using Lopital's rule) $= x_1 f(x_1) \Rightarrow$ (B)]

60 [Sol. $I = \int_{-\pi/4}^{\pi/4} \ln(\cos x+\sin x)\, dx$

$I = \int_{-\pi/4}^{\pi/4} \ln(\cos x-\sin x)\, dx$ hence $2I = \int_{-\pi/4}^{\pi/4} \ln(\cos 2x)\, dx$

$= \int_0^{\pi/2} \cos t\, dt = -\frac{\pi}{2}\ln 2 \Rightarrow I = -\frac{\pi}{4}\ln 2$]

61 [Sol. $f(x) = \cos(\tan^{-1}x)$

$f'(x) = -\frac{\sin(\tan^{-1}x)}{1+x^2}$

$I = \int_0^1 x\, f''(x)\, dx = x\, f'(x)\Big]_0^1 - \int_0^1 f'(x)\, dx$

$= f'(1) - [f(x)]_0^1 = f'(1) - [f(1) - f(0)] = f'(1) - f(1) + f(0)$

$f(0) = 1$; $f'(1) = -\frac{1}{2\sqrt{2}}$; $f(1) = \frac{1}{\sqrt{2}}$]

62 [Hint: note that $\sec^{-1}\sqrt{1+x^2} = \tan^{-1}x$; $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1}x$ for $x > 0$

$$I = \int \frac{e^{\tan^{-1}x}}{1+x^2}\left((\tan^{-1}x)^2 + 2\tan^{-1}x\right)dx \qquad \text{put } \tan^{-1}x = t$$

$$= \int e^t(t^2+2t)\,dt = e^t \cdot t^2 = e^{\tan^{-1}x}\left(\tan^{-1}x\right)^2 + C\ ]$$

63 [Hint: $I = \int_1^2 1.(lnx)^2\,dx = ln^2 x.x\Big]_1^2 - 2\int_1^2 \frac{lnx}{x}.x\ dx = 2\ ln^2 2 - 2\left[\int_1^2 ln\,x\ dx\right]$

$= 2\ ln^2 2 - 2\left[x\,lnx - x\right]_1^2 = 2\ ln^2 2 - 2\ [\ (2\ ln\ 2 - 2)\ (0 - 1)\ )\ ]$

$= 2\ ln^2 2 - 2\ [\ 2\ ln\ 2 - 1] = 2\ ln^2 2 - 4\ ln\ 2 + 2 = 2\ [\ ln^2 2 - 2\ ln\ 2 + 1\ ] = 2\left(ln\frac{2}{e}\right)^2 \Rightarrow$ (B)]

64 [Sol. Given $U_n = \int_0^1 x^n.(2-x)^n\ dx$; $V_n = \int_0^1 x^n.(1-x)^n\ dx$

in U_n put $x = 2t \Rightarrow dx = 2dt$

$\therefore \quad U_n = 2\int_0^{1/2} 2^n.t^n\,2^n\,(1-t)^n\ dt \qquad ....(1)$

Now $V_n = 2\int_0^{1/2} x^n\ (1-x)^n\ dx$ (Using Queen) .....(2)

From (1) and (2)

$U_n = 2^{2n}.\ V_n \Rightarrow$ (C)]

65 [Hint: $S'(x) = l\,n\,x^3.\ 3\,x^2 - l\,n\,x^2.\ 2\,x \qquad = 9\,x^2\,l\,n\,x - 4\,x\,l\,n\,x$

$= x\,l\,n\,x\,(9\,x - 4)$. Hence $\frac{S'(x)}{x} = l\,n\,x\,(9\,x-4)$.

Now it is obvious that $\frac{S'(x)}{x}$ is continuous and derivable in its domain.]

66 [Hint: using L Hospital's rule

$$l = \lim_{x\to 0}\frac{-x\sin x}{2-2\cos 2x} = \lim_{x\to 0}\frac{-x\sin x}{2(2\sin^2 x)} = \lim_{x\to 0}\frac{-1}{4\frac{\sin x}{x}} = -\frac{1}{4}\ ]$$

67 [Hint: LHS = secx + cosecx $= 2\sqrt{2} \Rightarrow x = \frac{\pi}{4}$ and $\frac{11\pi}{12}$]

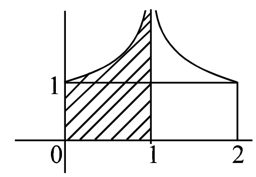
68 [Hint: $\displaystyle \lim_{n\to\infty} \frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots\ldots\ldots\ldots\ldots\ldots+\sqrt{n}}{n\sqrt{n}} = \int_0^1 \sqrt{x}\,dx = \frac{2}{3}$

$\therefore \quad S_n = \frac{2}{3}n^{3/2}$]

69 [Sol. $\displaystyle \int_0^2 \frac{dx}{(1-x)^2} = \int_0^{1^-} \frac{dx}{(1-x)^2} + \int_{1^+}^2 \frac{dx}{(1-x)^2}$

$\displaystyle = \left.\frac{1}{1-x}\right]_0^{1^-} + \left.\frac{1}{1-x}\right]_{1^+}^2$

$= (\infty - 1) + (-1) - (-\infty) \Rightarrow$ indeterminant



Note that the shaded area is divergent]

70 [Hint: $\displaystyle I = \int_0^{\pi/2} \frac{\sin x \cos x}{x\left(\frac{\pi}{2}-x\right)}dx = \int_0^{\pi/2} \frac{\sin 2x}{x(\pi-2x)}dx$; put $2x = t$

$\displaystyle I = \int_0^{\pi} \frac{\sin t}{t(\pi-t)}dt = \frac{1}{\pi}\int_0^{\pi}\left(\frac{\sin t}{t} + \frac{\sin t}{(\pi-t)}\right)dt = \frac{1}{\pi}\int_0^{\pi}\frac{\sin t}{t}dt + \frac{1}{\pi}\int_0^{\pi}\frac{\sin t}{\pi-t}dt$

$\displaystyle = \frac{1}{\pi}\int_0^{\pi}\frac{\sin t}{t}dt + \frac{1}{\pi}\int_0^{\pi}\frac{\sin t}{t}dt = \frac{2}{\pi}\int_0^{\pi}\frac{\sin t}{t}dt$ Ans.]