Complex Numbers

Exercise – 3

Q.1

$x+1=\sqrt{2}i \Rightarrow (x+1)^4 = 4$ Or $x^4+4x^3+6x^2+4x+9=12.$

Q.2

$a=\dfrac{1+i}{\sqrt{2}} \Rightarrow a^2 = i$, hence $a^6+a^4+a^2+1=i^6+i^4+i^2+1=0.$

Q.3

Let $Z_1 = a+i \;\&\; Z_2 = b+i$, then $\left(1+a^2\right)\left(1+b^2\right)=|Z_1|^2|Z_2|^2$

$\Rightarrow \left(1+a^2\right)\left(1+b^2\right)=\left(Z_1\overline{Z}_1\right)\left(Z_2\overline{Z}_2\right)=\left(Z_1\overline{Z}_2\right)\left(Z_2\overline{Z}_1\right)$

$\Rightarrow \left(1+a^2\right)\left(1+b^2\right)=\left(ab+1+(b-a)i\right)\left(ab+1-(b-a)i\right)$

Or $\left(1+a^2\right)\left(1+b^2\right)=(ab+1)^2+(b-a)^2.$

Q.4

Let $\sqrt{7-24i}=a+bi$, then $7-24i=a^2-b^2+2abi$

$\Rightarrow a^2-b^2=7 \;\&\; 2ab=-24.$

Now $\left(a^2+b^2\right)^2=\left(a^2-b^2\right)^2+4a^2b^2 \Rightarrow a^2+b^2=25.$

From $a^2+b^2=25 \;\&\; a^2-b^2=7$, we get $a^2=16 \;\&\; b^2=9.$

As $ab<0$, hence $a=4 \;\&\; b=-3$ or $a=-4 \;\&\; b=3.$

Required square roots are $4-3i \;\&\; -4+3i.$

Q.5

$Z_1=3i \Rightarrow \arg(Z_1)=\dfrac{\pi}{2} \;\&\; Z_2=-1-i \Rightarrow \arg(Z_2)=-\dfrac{3\pi}{4}.$

Now $\arg\left(\dfrac{Z_1}{Z_2}\right)=\arg(Z_1)-\arg(Z_2)=\dfrac{5\pi}{4}$

Now $\dfrac{5\pi}{4}$ is an angle in third quadrant, hence principal argument will be $-\dfrac{3\pi}{4}.$

Q.6

Let $Z=x+yi$, then $Z^2+|Z|=0\Rightarrow x^2-y^2+\sqrt{x^2+y^2}+2ixy=0$, hence $xy=0$ & $x^2-y^2+\sqrt{x^2+y^2}=0.$

$y=0\Rightarrow |x|=-x^2$, which is not possible as $x\in R$ and $x=0$ has already been considered.

$x=0\Rightarrow |y|=y^2\Rightarrow y=-1,1$ or 0.

Required solution set is $Z=0,-i,i.$

Q.7

Let $Z=x+yi$, then $Z^2+\bar{Z}=0\Rightarrow x^2-y^2+x+(2xy-y)i=0$

Hence $2xy-y=0$ & $x^2-y^2+x=0.$

$y=0\Rightarrow x^2+x=0$, which gives $x=0,-1$

$x=\frac{1}{2}\Rightarrow y^2=\frac{3}{4}$, which gives $y=-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}.$

Required solution set is $Z=0,-1,\frac{1-\sqrt{3}i}{2},\frac{1+\sqrt{3}i}{2}.$

Q.8

Let $Z=x+yi$, then $Z^2=\left(i\bar{Z}\right)^2\Rightarrow x^2-y^2+2xyi=y^2-x^2+2xyi$

Hence $x^2-y^2=0\Rightarrow x=y$ or $x=-y.$

Hence $Z=x+xi$ or $x-xi.$

Q.9

Let $Z=x+yi$, then $|Z+1|=Z+2+2i\Rightarrow\sqrt{(x+1)^2+y^2}=(x+2)+(y+2)i$

Hence $\sqrt{(x+1)^2+y^2}=(x+2)$ & $y+2=0.$

$y=-2$ gives $(x+2)^2-(x+1)^2=4\Rightarrow x=\frac{1}{2}.$

Required solution is $Z=\frac{1}{2}-2i.$

Q.10

Let $Z=x+yi$, then $i\bar{Z}=Z^2\Rightarrow y+xi=x^2-y^2+2xyi$, hence $x^2-y^2=y$ & $2xy=x.$

$2xy=x\Rightarrow x=0$ or $y=\frac{1}{2}$. Now $y=\frac{1}{2}$ gives $x=\pm\frac{\sqrt{3}}{2}$ & $x=0$ gives $y^2=-y$ i.e. $y=-1.$

Required solutions are $Z=-i,\ \frac{1\pm\sqrt{3}}{2}.$

Q.11

Let $Z = x + yi$, then $2|Z|^2 + Z^2 - 5 + \sqrt{3}i \Rightarrow 3x^2 + y^2 + 2xyi = 5 - \sqrt{3}i$

Hence $3x^2 + y^2 = 5$ & $2xy = -\sqrt{3}$.

Now $\left(3x^2 - y^2\right)^2 = \left(3x^2 + y^2\right)^2 - 12x^2y^2 \Rightarrow 3x^2 - y^2 = 4$ or -4.

$3x^2 + y^2 = 5$ & $3x^2 - y^2 = 4$ gives $x^2 = \frac{3}{2}$ & $y^2 = \frac{1}{2} \Rightarrow Z = \pm\frac{\sqrt{3} - i}{2}$ (as $xy < 0$)

$3x^2 + y^2 = 5$ & $3x^2 - y^2 = -4$ gives $x^2 = \frac{1}{6}$ & $y^2 = \frac{9}{2} \Rightarrow Z = \pm\left(\frac{1 - 3\sqrt{3}}{\sqrt{6}}\right)$ (as $xy < 0$).

Q.12

(i) Let $A(-6,0)$ & $B(0,-4)$ be two points on argand plane & $P(Z)$ be a moving a point.

Now $|Z + 6| = PA$ & $|Z + 4i| = PB$.

$\left|\frac{Z+6}{Z+4i}\right| = \frac{5}{3} \Rightarrow \frac{PA}{PB} = \frac{5}{3}$, hence P will trace a circle.

(ii) Let $A(-2,0)$ & $B(-4,0)$ be two points on argand plane & $P(Z)$ be a moving a point.

Now $|Z + 2| = PA$ & $|Z + 4| = PB$.

$\left|\frac{Z+2}{Z+4}\right| = 1 \Rightarrow PA = PB$, hence P will trace a straight line.

Q.13

Let $Z = x + yi$, then $Z + c|Z + 1| + i = 0 \Rightarrow x + iy + c\sqrt{(x+1)^2 + y^2} + i = 0$

Hence $c\sqrt{(x+1)^2 + y^2} = -x$ & $y = -1$. Now $y = -1$ gives $c^2(x+1)^2 = x^2 - c^2$

or $\left(c^2 - 1\right)x^2 + 2c^2x + 2c^2 = 0 \Rightarrow x = \frac{-c^2 \pm \sqrt{2c^2 - c^4}}{c^2 - 1}$, where $1 < c \le \sqrt{2}$. For $c = 1$, we get $x = -1$.

Hence $Z = \frac{-c^2 \pm \sqrt{2c^2 - c^4}}{c^2 - 1} - i$ for $1 < c \le \sqrt{2}$ & $Z = -1 - i$ for $c = 1$.

Q.14

Let $Z = x + yi$, then $Z^2 = \overline{Z} \Rightarrow x^2 - y^2 + 2xyi = x - iy$. Hence $x^2 - y^2 = x$ & $2xy = -y$.

Now $y = 0$ gives $x = 0$ & 1 & $x = -\frac{1}{2}$ gives $y = \pm\frac{\sqrt{3}}{2}$

Hence $Z = 0, 1, \frac{-1 \pm \sqrt{3}i}{2}$.

Q.15

$$\arg\left(\frac{3Z-6-3i}{2Z-8-6i}\right)=\frac{\pi}{4}\Rightarrow\arg\left(\frac{Z-2-i}{Z-4-3i}\right)=\frac{\pi}{4}$$

Now let $A(2,1)\,\&\,B(4,3)$ be two fixed point & $P(Z)$, then

$\arg\left(\frac{Z-2-i}{Z-4-3i}\right)=$ angle between $PA\,\&\,PB$.

As AB subtends a constant acute angle at P, hence locus of P will be major arc of a circle passing through $A\,\&\,B$.

Q.16

$$|Z+6|=|2Z+3|\Rightarrow(x+6)^2+y^2=(2x+3)^2+4y^2 \text{ or } x^2+y^2=9.$$

Q.17

Let $Z=\cos\alpha+i\sin\alpha$, then $Z^2+\overline{Z}=\cos 2\alpha+\cos\alpha+i(\sin 2\alpha-\sin\alpha)$

Or $Z^2+\overline{Z}=2\cos\frac{3\alpha}{2}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)$

Now $\arg\left(Z^2+\overline{Z}\right)=\frac{\alpha}{2}=\frac{1}{2}\arg(Z)$.

Q.18

(i) $\prod_{r=1}^{n}(a_r+ib_r)=x+iy \Rightarrow \prod_{r=1}^{n}|a_r+ib_r|^2=x^2+y^2$ or $\prod_{r=1}^{n}\left(a_r{}^2+b_r{}^2\right)=x^2+y^2$.

(ii) $\arg\left\{\prod_{r=1}^{n}(a_r+ib_r)\right\}=\sum_{r=1}^{n}\arg(a_r+ib_r)\Rightarrow\sum_{r=1}^{n}\tan^{-1}\left(\frac{b_r}{a_r}\right)=\tan^{-1}\left(\frac{y}{x}\right)$.

Q.19

$|Z_r|=1\Rightarrow Z_r=\frac{1}{\overline{Z_r}}$. Also $|Z|=|\overline{Z}|$.

Now $|Z_1+Z_2+\ldots+Z_n|=\left|\frac{1}{\overline{Z_1}}+\frac{1}{\overline{Z_2}}+\ldots+\frac{1}{\overline{Z_n}}\right|=\left|\frac{1}{Z_1}+\frac{1}{Z_2}+\ldots+\frac{1}{Z_n}\right|$.

Q.20

$$iZ^3+Z^2-Z+i=0\Rightarrow(iZ+1)\left(Z^2+i\right)=0$$

Hence $Z=i,\pm\sqrt{-i}\Rightarrow|Z|=1$.

Q.21

Let affixes of the points P, Q & O on argand plane be Z_1, Z_2 & 0.

Now $|Z_1+Z_2|^2 = |Z_1|^2 + |Z_2|^2 + 2\cos\theta$, where $\theta = \angle POQ = \arg\left(\frac{OP}{OQ}\right) = \arg\left(\frac{Z_1}{Z_2}\right)$.

Clearly if $|Z_1+Z_2|^2 = |Z_1|^2 + |Z_2|^2$, then $\theta = \frac{\pi}{2}$.

Q.22

$|Z_1+Z_2|^2 = |Z_1+Z_2||\overline{Z}_1+\overline{Z}_2| = |Z_1|^2 + |Z_2|^2 + Z_1\overline{Z}_2 + \overline{Z}_1 Z_2$ &

$|Z_1-Z_2|^2 = |Z_1-Z_2||\overline{Z}_1-\overline{Z}_2| = |Z_1|^2 + |Z_2|^2 - Z_1\overline{Z}_2 - \overline{Z}_1 Z_2$

$\Rightarrow |Z_1+Z_2|^2 + |Z_1-Z_2|^2 = 2|Z_1|^2 + 2|Z_2|^2$.

Q.23

By $|Z_1-Z_2|^2 = |Z_1|^2 + |Z_2|^2 - Z_1\overline{Z}_2 - \overline{Z}_1 Z_2$,

$|1-\overline{Z}_1 Z_2|^2 = 1 + |\overline{Z}_1 Z_2|^2 - Z_1\overline{Z}_2 - \overline{Z}_1 Z_2$ or $|1-\overline{Z}_1 Z_2|^2 = 1 + |Z_1|^2 |Z_2|^2 - Z_1\overline{Z}_2 - \overline{Z}_1 Z_2$

Now $|1-\overline{Z}_1 Z_2|^2 - |Z_1-Z_2|^2 = 1 + |Z_1|^2|Z_2|^2 - |Z_1|^2 - |Z_2|^2$

$\Rightarrow |1-\overline{Z}_1 Z_2|^2 - |Z_1-Z_2|^2 = \left(1-|Z_1|^2\right)\left(1-|Z_2|^2\right)$.

Q.24

$|iZ+3-5i| = |Z-5-3i|$, Now using $|Z_1+Z_2| \le |Z_1| + |Z_2|$ we get

$|Z-5-3i| \le |Z-1| + |4+3i| \Rightarrow |Z-5-3i| < 8$.

Q.25

$|(1+i)Z^3 + iZ| \le |Z^3| + |iZ^3| + |iZ| \Rightarrow |(1+i)Z^3 + iZ| \le 2|Z|^3 + |Z|$

Now $|Z| < \frac{1}{2} \Rightarrow |(1+i)Z^3 + iZ| < \frac{3}{4}$.

Q.26

$Z^2 + \alpha Z + \beta = 0 \ldots (i) \Rightarrow \overline{Z}^2 + \overline{\alpha}\,\overline{Z} + \overline{\beta} = 0$.

As Z is real, $Z = \overline{Z}$, hence $Z^2 + \overline{\alpha} Z + \overline{\beta} = 0 \ldots (ii)$

Applying the condition of common root on $(\text{i})\&(\text{ii})$ gives

$$(\bar{\alpha}-\alpha)(\alpha\bar{\beta}-\bar{\alpha}\beta)=(\beta-\bar{\beta})^2 \text{ or } \frac{(\bar{\beta}-\beta)^2}{(\alpha-\bar{\alpha})^2}+\frac{(\bar{\beta}-\beta)}{(\alpha-\bar{\alpha})}+\beta=0.$$

Q.27

$$Z^2+aZ+b=0 \Rightarrow \left(Z+\frac{a}{2}\right)^2=\frac{a^2}{4}-b.$$

Clearly if $\frac{a^2}{4}-b\geq 0$, then Z must be real, hence for nonreal complex roots $a^2<4b$.

Q.28

By properties of modulus, $\left||Z|-\frac{1}{|Z|}\right| \leq \left|Z+\frac{1}{Z}\right| \leq |Z|+\frac{1}{|Z|} \Rightarrow |Z|+\frac{1}{|Z|}\geq 2 \ \& \ \left||Z|-\frac{1}{|Z|}\right|\leq 2.$

Now first inequality is by itself true as $|Z|+\frac{1}{|Z|}\geq 2\sqrt{|Z|\times\frac{1}{|Z|}}$ by A.M. $\geq$ G.M.

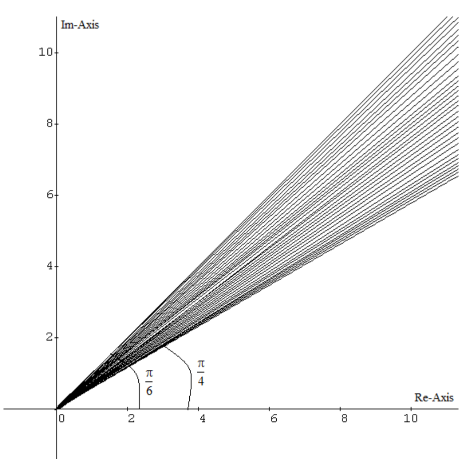
$$\left||Z|-\frac{1}{|Z|}\right|\leq 2 \Rightarrow -2\leq |Z|-\frac{1}{|Z|}\leq 2.$$

(i) $|Z|-\frac{1}{|Z|}\leq 2 \Rightarrow |Z|^2-2|Z|-1\leq 0$ or $|Z|\leq 1+\sqrt{2}$.

(ii) $|Z|-\frac{1}{|Z|}\geq -2 \Rightarrow |Z|^2+2|Z|-1\geq 0$ or $|Z|\geq \sqrt{2}-1$.

Hence $\sqrt{2}-1\leq |Z|\leq 1+\sqrt{2}$.

Q.29



Q.30

$x^2-2x+1=-3\Rightarrow(x-1)^2=\sqrt{3}i$ or $x=1\pm\sqrt{3}i.$

Hence $x=2e^{\pm\frac{i\pi}{3}}$, which implies $Z_1^{\,n}=2^n\left(\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}\right)$ & $Z_2^{\,n}=2^n\left(\cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3}\right)$

Now $Z_1^{\,n}+Z_2^{\,n}=2^n\left(2\cos\frac{n\pi}{3}\right).$

Q.31

Let A & B be the points $(-1,0)$&$(1,0)$ & $P(Z)$ be a moving point, then

$|Z+1|^2+|Z+1|^2=2^2\Rightarrow PA^2+PB^2=AB^2.$

Hence P lies on the circle having AB as diameter.

Q.32

Let $Z_k=\cos\theta_k+i\sin\theta_k$, then $\sum_{k=1}^{3}\cos\theta_k=0=\sum_{k=1}^{3}\sin\theta_k\Rightarrow Z_1+Z_2+Z_3=0.$

(i) $Z_1+Z_2+Z_3=0\Rightarrow Z_1^{\,2}+Z_2^{\,2}+Z_3^{\,2}=(Z_1+Z_2+Z_3)^2-2(Z_1Z_2+Z_2Z_3+Z_3Z_1).$

Now $Z_1Z_2+Z_2Z_3+Z_3Z_1=Z_1Z_2Z_3\left(\frac{1}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3}\right)$

$\therefore$ $Z_1Z_2+Z_2Z_3+Z_3Z_1=Z_1Z_2Z_3\left(\overline{Z}_1+\overline{Z}_2+\overline{Z}_3\right)=0$ as $|Z_k|=1$ and hence $\overline{Z}_k=\frac{1}{Z_k}.$

$\Rightarrow Z_1^{\,2}+Z_2^{\,2}+Z_3^{\,2}=0$ or $\sum_{k=1}^{3}(\cos 2\theta_k+i\sin 2\theta_k)=0.$

Hence $\cos 2\theta_1+\cos 2\theta_2+\cos 2\theta_3=0=\sin 2\theta_1+\sin 2\theta_2+\sin 2\theta_3$

(ii) $Z_1+Z_2+Z_3=0\Rightarrow Z_1^{\,3}+Z_2^{\,3}+Z_3^{\,3}=3Z_1Z_2Z_3.$

$\therefore\sum_{k=1}^{3}(\cos 3\theta_k+i\sin 3\theta_k)=3\cos(\theta_1+\theta_2+\theta_3)+3i\sin(\theta_1+\theta_2+\theta_3)$

Hence $\cos 3\theta_1+\cos 3\theta_2+\cos 3\theta_3=3\cos(\theta_1+\theta_2+\theta_3)$

Q.33

Let $\frac{1}{Z+a}+\frac{1}{Z+b}+\frac{1}{Z+c}+\frac{1}{Z+d}=\frac{2}{Z}$, then $\frac{1}{\overline{Z}+a}+\frac{1}{\overline{Z}+b}+\frac{1}{\overline{Z}+c}+\frac{1}{\overline{Z}+d}=\frac{2}{\overline{Z}}$

Clearly if ω is a root, then $\overline{\omega}$ i.e. ω^2 must also be a root as $a,b,c,d\in R.$

$\therefore\frac{1}{a+\omega^2}+\frac{1}{b+\omega^2}+\frac{1}{c+\omega^2}+\frac{1}{d+\omega^2}=\frac{2}{\omega^2}\quad ...(i)$

Now $(Z+b)(Z+c)(Z+d)Z+(Z+a)(Z+c)(Z+d)Z+(Z+a)(Z+b)(Z+d)Z$
$+(Z+a)(Z+b)(Z+c)Z=2(Z+a)(Z+b)(Z+c)(Z+d)$
or $2Z^4+(a+b+c+d)Z^3-(abc+bcd+cda+dab)Z-2abcd=0$

Let the roots be $\omega, \omega^2, \alpha \,\&\, \beta$, then
$\omega.\omega^2+\omega^2.\alpha+\alpha.\beta+\omega.\alpha+\omega.\beta+\omega^2.\beta=0 \Rightarrow 1+\alpha.\beta-\alpha-\beta=0.$
Hence 1 must also be a root $(\text{let } \beta=1)$.
$\therefore \frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1}=2 \;\; ...(iv)$
Further $\omega.\omega^2.\alpha.\beta=-abcd \Rightarrow \alpha=-abcd$
& $\omega+\omega^2+\alpha+\beta=-\frac{a+b+c+d}{2} \Rightarrow \alpha=-\frac{a+b+c+d}{2}$
$\therefore a+b+c+d=2abcd \;\; ...(iii)$
Lastly $\omega.\omega^2.\alpha+\omega^2.\alpha.\beta+\alpha.\beta.\omega+\beta.\omega.\omega^2=-\frac{abc+bcd+cda+dab}{2} \Rightarrow abc+bcd+cda+dab=2 \;\; ...(ii)$

Q.34

(i)

$(1+x)^n=a_0+a_1x+a_2x^2+a_3x^3+...+a_nx^n$, then by substituting $i \,\&\, -i$,
$(1+i)^n=a_0+a_1i-a_2-a_3i+a_4+a_5i-a_6-a_7i+...$
$(1-i)^n=a_0-a_1i-a_2+a_3i+a_4-a_5i-a_6+a_7i+...$
Adding the two gives $2(a_0-a_2+a_4-a_6+...)=(1+i)^n+(1-i)^n$.
Now $1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) \Rightarrow (1+i)^n+(1-i)^n=2\left(\sqrt{2}\right)^n\cos\frac{n\pi}{4}$.
Hence $a_0-a_2+a_4-a_6+...=2^{n/2}\cos\frac{n\pi}{4}$.

(ii)

$(1+x)^n=a_0+a_1x+a_2x^2+a_3x^3+...+a_nx^n$, then by substituting $i \,\&\, -i$,
$(1+i)^n=a_0+a_1i-a_2-a_3i+a_4+a_5i-a_6-a_7i+...$
$(1-i)^n=a_0-a_1i-a_2+a_3i+a_4-a_5i-a_6+a_7i+...$

Subtracting the two gives $2i(a_1 - a_3 + a_5 - a_7 + \ldots) = (1+i)^n + (1-i)^n$.

Now $1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \Rightarrow (1+i)^n - (1-i)^n = 2\left(\sqrt{2}\right)^n i\sin\frac{n\pi}{4}$.

Hence $a_1 - a_3 + a_5 - a_7 + \ldots = 2^{n/2}\sin\frac{n\pi}{4}$.

(iii)

Now $\left(a_0 - a_2 + a_4 - a_6 + \ldots\right)^2 + \left(a_1 - a_3 + a_5 - a_7 + \ldots\right)^2 = 2^n$.

(iv)

$(1+x)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$, then by substituting 1, ω & ω^2,

$2^n = a_0 + a_1 + a_2 + a_3 + \ldots$

$(1+\omega)^n = a_0 + a_1\omega + a_2\omega^2 + a_3 + a_4\omega + a_5\omega^2 + a_6 + \ldots$

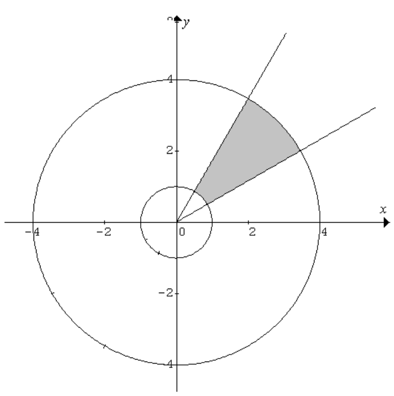
$\left(1+\omega^2\right)^n = a_0 + a_1\omega^2 + a_2\omega + a_3 + a_4\omega^2 + a_5\omega + a_6 + \ldots$

Adding the three relations gives $3(a_0 + a_3 + a_6 + \ldots) = 2^n + \left(-\omega^2\right)^n + (-\omega)^n$

Now $-\omega = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3} \Rightarrow \left(-\omega^2\right)^n + (-\omega)^n = 2\cos\frac{n\pi}{3}$.

Hence $a_0 + a_3 + a_6 + \ldots = \dfrac{2^n + 2\cos\frac{n\pi}{3}}{3}$.

Q.35



Required area is area of shaded sector as shown in the figure.

Area of sector of angle $\frac{\pi}{6}$ of circle of radius 4 will be $\frac{4\pi}{3}$.

Area of sector of angle $\frac{\pi}{6}$ of circle of radius 1 will be $\frac{\pi}{12}$.

Hence required area $= \frac{4\pi}{3} - \frac{\pi}{12}$ i.e. $\frac{5\pi}{4}$.

Q.36

$|Z_1 - Z_2|^2 = |Z_1|^2 + |Z_2|^2 - \left(Z_1 \overline{Z}_2 + Z_2 \overline{Z}_1\right)$

or $|Z_1 - Z_2|^2 = |Z_1|^2 + |Z_2|^2 - 2|Z_1||Z_2|\cos\theta$, where $\theta = \arg(Z_1) - \arg(Z_2)$

$\Rightarrow |Z_1 - Z_2|^2 = |Z_1|^2 + |Z_2|^2 - 2|Z_1||Z_2| + 2|Z_1||Z_2|(1-\cos\theta)$

$\Rightarrow |Z_1 - Z_2|^2 = \left(|Z_1| - |Z_2|\right)^2 + 4|Z_1||Z_2|\sin^2\frac{\theta}{2}.$

Now $|\sin\theta| \le |\theta| \Rightarrow \sin^2\frac{\theta}{2} \le \frac{\theta^2}{4}$, hence

$|Z_1 - Z_2|^2 \le \left(|Z_1| - |Z_2|\right)^2 + \theta^2$ or $|Z_1 - Z_2|^2 \le \left(|Z_1| - |Z_2|\right)^2 + \left(\arg(Z_1) - \arg(Z_2)\right)^2.$

Q.37

$Z^7 + 4Z^3 + 11 = 0 \Rightarrow \left|Z^7 + 4Z^3\right| = 11.$

But $\left||Z|^7 - 4|Z|^3\right| \le \left|Z^7 + 4Z^3\right| \le |Z|^7 + 4|Z|^3$ or $|Z|^7 + 4|Z|^3 \ge 11$ & $\left||Z|^7 - 4|Z|^3\right| \le 11.$

Now if $|Z| \le 1$, then $|Z|^7 + 4|Z|^3 \le 5$, hence for all Z, $|Z| > 1$

& if $|Z| \ge 2$, then $\left||Z|^7 - 4|Z|^3\right| \le 11$ is invalid, hence for all Z, $|Z| < 2$.

All 7 roots of given equation lie in $1 < |Z| < 2$.

Q.38

Let $Z_1 = \cos A + i\sin A, Z_2 = \cos B + i\sin B$ & $Z_1 = \cos C + i\sin C$, then

$\sum \cos A = -\frac{3}{2}$ & $\sum \sin A = \frac{3\sqrt{3}}{2} \Rightarrow Z_1 + Z_2 + Z_3 = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i$

Now $|Z_1 + Z_2 + Z_3| \le |Z_1| + |Z_2| + |Z_3| \Rightarrow |Z_1 + Z_2 + Z_3| \le 3.$

But $|Z_1 + Z_2 + Z_3| = \sqrt{\frac{9}{4} + \frac{27}{4}} = 3.$

Hence $A = B = C = \frac{2\pi}{3}$ $\therefore$ $\tan A + \tan B + \tan C = -3\sqrt{3}.$

Q.39

$Z^5 - 1 = (Z-1)\left(Z^4 + Z^3 + Z^2 + Z + 1\right)$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the roots of

$Z^4 + Z^3 + Z^2 + Z + 1 = 0$. Now by factor theorem

$(Z - \alpha_1)(Z - \alpha_1)(Z - \alpha_1)(Z - \alpha_1) = Z^4 + Z^3 + Z^2 + Z + 1$

Substituting $Z=\omega$ gives

$$(\omega-\alpha_1)(\omega-\alpha_1)(\omega-\alpha_1)(\omega-\alpha_1)=\omega^4+\omega^3+\omega^2+\omega+1=-\omega^2$$

similarly Substituting $Z=\omega^2$ gives

$$\left(\omega^2-\alpha_1\right)\left(\omega^2-\alpha_1\right)\left(\omega^2-\alpha_1\right)\left(\omega^2-\alpha_1\right)=\omega^8+\omega^6+\omega^4+\omega^2+1=-\omega.$$

Hence $\dfrac{(\omega-\alpha_1)(\omega-\alpha_1)(\omega-\alpha_1)(\omega-\alpha_1)}{\left(\omega^2-\alpha_1\right)\left(\omega^2-\alpha_1\right)\left(\omega^2-\alpha_1\right)\left(\omega^2-\alpha_1\right)}=\omega.$

Q.40

Let α be a real root of $(1+2i)\alpha^3-2(3+i)\alpha^2+(5-4i)\alpha+2a^2=0\ \ ...(i)$

Taking conjugate gives $(1-2i)\alpha^3-2(3-i)\alpha^2+(5+4i)\alpha+2a^2=0\ \ ...(ii)$

Adding $(i)\&(ii)$ gives $\alpha^3-6\alpha^2+5\alpha+2a^2=0\ \ ...(iii)$

Subtracting (ii) from (i) gives $2\alpha^3-2\alpha^2-4\alpha=0\ \ ...(iv)$

Or $2\alpha\left(\alpha^2-\alpha-2\right)=0$ gives $\alpha=0,-1,2.$

For these values of α, from (iii) we get $a=0,\pm\sqrt{6}\ \&\ \pm\sqrt{3}.$

Q.41

$$\left(|Z_1|+|Z_2|\right)\left|\frac{Z_1}{|Z_1|}+\frac{Z_2}{|Z_2|}\right|=\left|Z_1+Z_2+\frac{|Z_2|}{|Z_1|}Z_1+\frac{|Z_1|}{|Z_2|}Z_2\right|$$

$$\Rightarrow\left(|Z_1|+|Z_2|\right)\left|\frac{Z_1}{|Z_1|}+\frac{Z_2}{|Z_2|}\right|\le|Z_1+Z_2|+\left|\frac{|Z_2|}{|Z_1|}Z_1+\frac{|Z_1|}{|Z_2|}Z_2\right|$$

Now $\left|\dfrac{|Z_2|}{|Z_1|}Z_1+\dfrac{|Z_1|}{|Z_2|}Z_2\right|=|Z_1+Z_2|$, hence

$$\left(|Z_1|+|Z_2|\right)\left|\frac{Z_1}{|Z_1|}+\frac{Z_2}{|Z_2|}\right|\le 2|Z_1+Z_2|.$$

Q.42

$$|Z-1|=\big|Z-|Z|+|Z|-1\big|\Rightarrow|Z-1|\le\big||Z|-1\big|+\big|Z-|Z|\big|$$

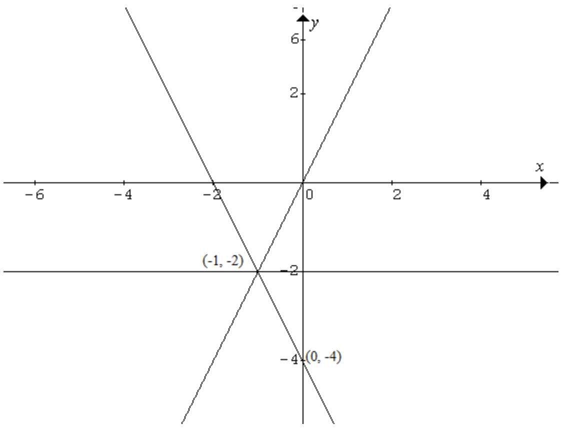
$$\Rightarrow|Z-1|\le\big||Z|-1\big|+|Z|\left|\frac{Z}{|Z|}-1\right|$$

Now let $\frac{Z}{|Z|} = \cos\theta + i\sin\theta$, then $\frac{Z}{|Z|} - 1 = \cos\theta - 1 + i\sin\theta$ or $\frac{Z}{|Z|} - 1 = 2\sin\frac{\theta}{2}\left(-\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}\right)$.

Hence $\left|\frac{Z}{|Z|} - 1\right| = \left|2\sin\frac{\theta}{2}\right|$.

Now for $|\sin\theta| \le |\theta| \Rightarrow \left|2\sin\frac{\theta}{2}\right| \le |\theta|$, therefore $|Z-1| \le \left||Z|-1\right| + |Z||\theta|$ or $|Z-1| \le \left||Z|-1\right| + |Z||\arg(Z)|$.

Q.43



The line $(2+i)Z + (2-i)\overline{Z} = 0$ & $i\overline{Z} - iZ = 4$,

in $x - y$ coordinates are

$(2+i)(x+iy) + (2-i)(x-iy) = 0$ & $i(x+iy) - i(x-iy) = 4$

i.e. $2x - y = 0$ & $y = -2$.

Point of intersection : $(-1,-2)$

Also the first line passes through $(0,0)$ &

image of $(0,0)$ in $y = -2$ is $(0,-4)$.

Hence refleted line will be the line joining $(0,-4)$ & $(-1,-2)$ i.e. $y + 2x + 4 = 0$

Now put $x = \frac{Z+\overline{Z}}{2}$ & $y = \frac{Z-\overline{Z}}{2i}$ to get $\frac{Z-\overline{Z}}{2i} + 2\frac{Z+\overline{Z}}{2} + 4 = 0$

or $(2-i)Z + (2+i)\overline{Z} = 8$.

Q.44

Equation of AB, $\frac{Z-a}{\overline{Z}-\overline{a}} = \frac{b-a}{\overline{b}-\overline{a}}$ & that of CD, $\frac{Z-c}{\overline{Z}-\overline{c}} = \frac{d-c}{\overline{d}-\overline{c}}$.

Eliminating $\overline{Z}$ gives $(Z-a)\frac{\overline{b}-\overline{a}}{b-a} + \overline{a} = (Z-c)\frac{\overline{d}-\overline{c}}{d-c} + \overline{c}$.

Also $|a|=|b|=|c|=|d|=r \Rightarrow \bar{a}=\frac{r}{a}, \bar{b}=\frac{r}{b}, \bar{c}=\frac{r}{c}, \bar{d}=\frac{r}{d}$, hence

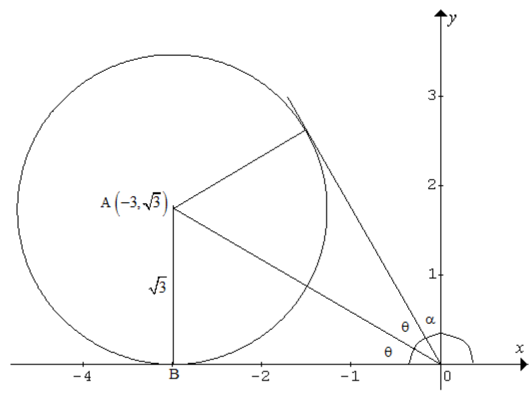
$$(Z-a)\frac{\bar{b}-\bar{a}}{b-a}+\bar{a}=(Z-c)\frac{\bar{d}-\bar{c}}{d-c}+\bar{c} \Rightarrow -(Z-a)\frac{1}{ab}+\frac{1}{a}=-(Z-c)\frac{1}{cd}+\frac{1}{c}$$

or $Z=\dfrac{\frac{1}{a}+\frac{1}{b}-\frac{1}{c}-\frac{1}{d}}{\frac{1}{ab}-\frac{1}{cd}}$.

Q.45

$\left|Z+3-\sqrt{3}i\right|=\sqrt{3}$ represents the circle having center at $\left(-3,\sqrt{3}\right)$ & radius $=\sqrt{3}$.

Let $P(Z)$ be any point on this circle, then least $\arg(z)=\theta$, where $\tan\theta$ is slope of that tangent from the origin to this circle which makes smallest angle with positive direction of real axis.



$OA=2\sqrt{3}$ & $AB=\sqrt{3}$, hence $\theta=\alpha=\frac{\pi}{6}$

& hence least $\arg(Z)=\frac{\pi}{2}+\frac{\pi}{6}$ or $\frac{2\pi}{3}$.

Also $|Z|=$ length of tangent to this circle from origin i.e. $\left|3-\sqrt{3}i\right|-\sqrt{3}$ or 3. Hence $Z=3e^{\frac{2i\pi}{3}}$

Q.46

$$\sin\frac{(2r-1)\pi}{14}=\sin\left(\pi-\frac{(2r-1)\pi}{14}\right) \text{ \& } \sin\frac{7\pi}{14}=1$$

gives $\prod_{r=1}^{7}\sin\frac{(2r-1)\pi}{14}=\sin^2\frac{\pi}{14}\sin^2\frac{3\pi}{14}\sin^2\frac{5\pi}{14}$ Or $\prod_{r=1}^{7}\sin\frac{(2r-1)\pi}{14}=\cos^2\frac{\pi}{7}\cos^2\frac{2\pi}{7}\cos^2\frac{4\pi}{7}$.

Now $\prod_{r=1}^{7}\sin\frac{(2r-1)\pi}{14}=\left(\frac{8\sin\frac{\pi}{7}\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}}{8\sin\frac{\pi}{7}}\right)^2$

$\Rightarrow\prod_{r=1}^{7}\sin\frac{(2r-1)\pi}{14}=\left(\frac{\sin\frac{8\pi}{7}}{8\sin\frac{\pi}{7}}\right)^2=\frac{1}{64}.$

Alternately

Let $Z=\cos\theta+i\sin\theta$, for $\theta=\frac{\pi}{7},\frac{2\pi}{7}\,\&\,\frac{4\pi}{7}$, then $Z^7=1$.

Or $(Z-1)\left(Z^6+Z^5+Z^4+Z^3+Z^2+Z+1\right)=0.$

Now roots of $Z^6+Z^5+Z^4+Z^3+Z^2+Z+1=0$ are $Z=e^{i\theta}$ for $\theta=\frac{\pi}{7},\frac{2\pi}{7},\frac{4\pi}{7},\frac{6\pi}{7},\frac{8\pi}{7},\frac{10\pi}{7}.$

Dividing by Z^3 gives $Z^3+\frac{1}{Z^3}+Z^2+\frac{1}{Z^2}+Z+\frac{1}{Z}+1=0$, roots of which are $\theta=\frac{\pi}{7},\frac{2\pi}{7},\frac{4\pi}{7}.$

Now $2\cos3\theta+2\cos2\theta+2\cos\theta+1=0$ or $8\cos^3\theta+4\cos^2\theta-4\cos\theta-1=0.$

Hence $\cos^2\frac{\pi}{7}\cos^2\frac{2\pi}{7}\cos^2\frac{4\pi}{7}=\frac{1}{64}.$

Q.47

Let $Z=\cos\theta+i\sin\theta$, then $\sum_{r=0}^{n}\cos(r\theta)=\text{Re}\left(\sum_{r=0}^{n}Z^r\right)$

Or $\sum_{r=0}^{n}\cos(r\theta)=\frac{\sum_{r=0}^{n}Z^r+\sum_{r=0}^{n}\overline{Z}^r}{2}\Rightarrow 2\sum_{r=0}^{n}\cos(r\theta)=\frac{Z^{n+1}-1}{Z-1}+\frac{\overline{Z}^{n+1}-1}{\overline{Z}-1}.$

$\Rightarrow 2\sum_{r=0}^{n}\cos(r\theta)=\frac{Z^n+\overline{Z}^n-Z^{n+1}-\overline{Z}^{n+1}-Z-\overline{Z}+2}{2-Z-\overline{Z}}$

$\Rightarrow\sum_{r=0}^{n}\cos(r\theta)=\frac{\cos n\theta-\cos(n+1)\theta-\cos\theta+1}{2-2\cos\theta}.$

$$\Rightarrow \sum_{r=0}^{n}\cos(r\theta)=\frac{\sin\frac{\theta}{2}-\sin\frac{(2n+1)\theta}{2}}{2\sin\frac{\theta}{2}}.$$

Q.48

Let $Z_k=\cos\theta_k+i\sin\theta_k$, then $\sum_{k=1}^{3}\cos\theta_k=0=\sum_{k=1}^{3}\sin\theta_k \Rightarrow Z_1+Z_2+Z_3=0.$

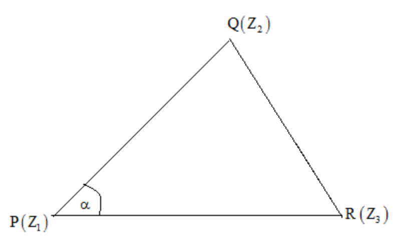
Now $Z_1+Z_2+Z_3=0 \Rightarrow Z_1^{\,3}+Z_2^{\,3}+Z_3^{\,3}=3Z_1Z_2Z_3.$

$\therefore \sum_{k=1}^{3}(\cos 3\theta_k+i\sin 3\theta_k)=3\cos(\theta_1+\theta_2+\theta_3)+3i\sin(\theta_1+\theta_2+\theta_3)$

Hence $\cos 3\theta_1+\cos 3\theta_2+\cos 3\theta_3=3\cos(\theta_1+\theta_2+\theta_3)$

& $\sin 3\theta_1+\sin 3\theta_2+\sin 3\theta_3=3\sin(\theta_1+\theta_2+\theta_3)$, where $\theta_1=\alpha,\theta_2=\beta,\theta_3=\gamma$.

Q.49



As $\angle PQR=\angle PRQ=\frac{1}{2}(\pi-\alpha)$, hence $PQ=PR$

i.e. $|Z_2-Z_1|=|Z_3-Z_1|$.

Now by Coni's theorem $\frac{Z_2-Z_1}{Z_3-Z_1}=\cos\alpha+i\sin\alpha.$

Or $\frac{Z_2-Z_1}{Z_3-Z_1}-\cos\alpha=i\sin\alpha \Rightarrow \left(\frac{Z_2-Z_1}{Z_3-Z_1}\right)^2-2\left(\frac{Z_2-Z_1}{Z_3-Z_1}\right)\cos\alpha+\cos^2\alpha=-\sin^2\alpha$

$\Rightarrow (Z_2-Z_1)^2-2(Z_2-Z_1)(Z_3-Z_1)\cos\alpha=(Z_3-Z_1)^2 \Rightarrow (Z_3-Z_2)^2=4(Z_3-Z_1)(Z_1-Z_2)\sin^2\frac{\alpha}{2}.$

Q.50

$PA=2PB \Rightarrow |Z-6i|=2|Z-3|$ or $(Z-6i)(\overline{Z}+6i)=4(Z-3)(\overline{Z}-3)$

$\Rightarrow Z\overline{Z}=(4+2i)Z+(4-2i)\overline{Z}$ ...(i)

Now Let $Z=x+iy$, then (i) gives $x^2+y^2-8x-4y=0$, which represents a circle.

Q.51

$|1-Z_1\overline{Z}_2|^2-|Z_1-Z_2|^2=(1-Z_1\overline{Z}_2)(1-\overline{Z}_1Z_2)-(Z_1-Z_2)(\overline{Z}_1-\overline{Z}_2).$

$$= 1 + |Z_1|^2 |Z_2|^2 - |Z_1|^2 - |Z_2|^2$$
$$= \left(1 - |Z_1|^2\right)\left(1 - |Z_2|^2\right).$$

Q.52

$$\frac{a-d}{b-c} + \frac{\bar{a}-\bar{d}}{\bar{b}-\bar{c}} = 0 \ \& \ \frac{b-d}{c-a} + \frac{\bar{b}-\bar{d}}{\bar{c}-\bar{a}} = 0 \Rightarrow \frac{c-d}{a-b} + \frac{\bar{a}-\bar{d}}{\bar{a}-\bar{b}} = 0.$$

Q.53

As given α^k is n^{th} root of unity for $k = 0, 1, 2, ..., n-1$, hence

$$(Z-1)(Z-\alpha)\left(Z-\alpha^2\right)...\left(Z-\alpha^{n-1}\right) = Z^n - 1$$

Or $\ln(Z-1) + \ln(Z-\alpha) + \ln\left(Z-\alpha^2\right) + ... + \ln\left(Z-\alpha^{n-1}\right) = \ln\left(Z^n - 1\right).$

Differentiating w.r.to Z gives

$$\frac{1}{Z-1} + \frac{1}{Z-\alpha} + \frac{1}{Z-\alpha^2} + ... + \frac{1}{Z-\alpha^{n-1}} = \frac{n Z^{n-1}}{Z^n - 1}.$$

Hence $\frac{1}{Z-1} + \frac{1}{Z-\alpha} + \frac{1}{Z-\alpha^2} + ... + \frac{1}{Z-\alpha^{n-1}} = 0 \Rightarrow Z^{n-1} = 0.$

Q.54

Let $Z_1 = \cos\alpha + i\sin\alpha, Z_2 = \cos\beta + i\sin\beta \ \& \ Z_3 = \cos\gamma + i\sin\gamma$, then

$Z_1 + Z_1Z_2 + Z_1Z_2Z_3 = 0.$

Or $Z_1\left(1 + Z_2 + Z_2Z_3\right) = 0 \Rightarrow 1 + Z_2 + Z_2Z_3 = 0.$

$\Rightarrow \cos\beta + \cos(\beta+\gamma) = -1 = \sin\beta + \sin(\beta+\gamma)$

Square and add to get

$2\cos\beta\cos(\beta+\gamma) + 2\sin\beta\sin(\beta+\gamma) = -1$ or $\cos\gamma = -\frac{1}{2}.$

Hence $\beta = \gamma = \frac{2\pi}{3}$ and $\tan\frac{\beta}{2} = \sqrt{3} \ \& \ \tan\gamma = -\sqrt{3}.$

Q.55

Let $P = \sin 1^o \sin 2^o \sin 3^o ... \sin 89^o$

$\sin(90-\theta) = \cos\theta \Rightarrow P = \left(\sin 1^0 \sin 2^o ... \sin 44^o\right)\sin 45^o \left(\cos 44^o ... \cos 2^0 \cos 1^o\right)$

Also $P = \left(\sin 1^o \sin 3^o \sin 5^o ... \sin 89^o\right)\left(\sin 2^o \sin 4^o \sin 6^o ... \sin 90^0\right)$

Or $2^{44}P = \frac{1}{\sqrt{2}}\left(\sin 2^o \sin 4^o \sin 6^o \ . \ . \ . \sin 88^o\right).$

Hence $P = \left(\sin 1^o \sin 3^o \sin 5^o ... \sin 89^o\right) 2^{44}\sqrt{2}\,P$ or $\sin 1^o \sin 3^o \sin 5^o ... \sin 89^o = \frac{1}{2^{44}\sqrt{2}}.$

Alternately

$$-1 = e^{i\pi} \Rightarrow (-1)^{1/90} = e^{\frac{(2k+1)\pi}{90}i} \ \forall \ k = 0,1,2,...,89 \Rightarrow Z^{90} + 1 = \prod_{k=0}^{44}\left(Z - e^{\frac{(2k+1)\pi}{90}i}\right)$$

Now multiplying conjugate pairs gives, $Z^{90} + 1 = \prod_{k=0}^{45}\left(Z^2 - \left(e^{\frac{(2k+1)\pi}{90}i} + e^{\frac{(179-2k)\pi}{90}i}\right)Z + 1\right)$

$$\Rightarrow Z^{90} + 1 = \prod_{k=0}^{45}\left(Z^2 - 2\cos\frac{(2k+1)\pi}{90}\,Z + 1\right) \ \Rightarrow Z^{45} + \frac{1}{Z^{45}} = \prod_{k=0}^{45}\left(Z + \frac{1}{Z} - 2\cos\frac{(2k+1)\pi}{90}\right)$$

Now for $Z = \cos 0 + i\sin 0$ we get $\prod_{k=0}^{45}\left(1 - \cos\frac{(2k+1)\pi}{90}\right) = \frac{1}{2^{44}}$

or $\prod_{k=0}^{45}\left(2\sin^2\frac{(2k+1)\pi}{180}\right) = \frac{1}{2^{44}} \Rightarrow \prod_{k=0}^{45}\left(\sin\frac{(2k+1)\pi}{180}\right) = \frac{1}{2^{44}\sqrt{2}}.$

Q.56

Let $aZ^3 + bZ^2 + cZ + d = a(Z-\alpha)(Z-\beta)(Z-\gamma)$

Now for $Z = i,\ -ai - b + ci + d = a(i-\alpha)(i-\beta)(i-\gamma)\ ...(i)$

and for $Z = -i,\ ai - b - ci + d = -a(i+\alpha)(i+\beta)(i+\gamma)\ ...(ii)$

Multiplying $(i)\,\&\,(ii)$ gives $(-ai - b + ci + d)(ai - b - ci + d) = a^2\left(1+\alpha^2\right)\left(1+\beta^2\right)\left(1+\gamma^2\right)$

or $\left(1+\alpha^2\right)\left(1+\beta^2\right)\left(1+\gamma^2\right) = \frac{(d-b)^2 + (c-a)^2}{a^2}.$

Q.57

Given $\left(1+x+x^2\right)^{3n} = \sum_{r=0}^{6n} a_r x^r$. Now recall that $1 + \omega^r + \omega^{2r} = \begin{cases} 3 & \text{if } r = 3m \\ 0 & \text{otherwise} \end{cases}.$

Substituting $1,\ \omega\,\&\,\omega^2$ in the given expansion and adding the three results gives

$$3^{3n} + \left(1+\omega+\omega^2\right)^{3n} + \left(1+\omega^2+\omega^4\right)^{3n} = \sum_{r=0}^{6n} a_r\left(1+\omega^r+\omega^{2r}\right)$$

or $3\left(a_0+a_3+a_6+...+a_{6n}\right)=3^{3n}.$

Smilarly

$$3^{3n}+\left(1+\omega+\omega^2\right)^{3n}\omega^2+\left(1+\omega^2+\omega^4\right)^{3n}\omega=\sum_{r=0}^{6n}a_r\left(1+\omega^{r+2}+\omega^{2r+1}\right)$$

or $3\left(a_1+a_4+a_7+...+a_{6n-2}\right)=3^{3n}.$

Q.58

$-1=e^{i\pi}\Rightarrow(-1)^{1/8}=e^{\frac{(2k+1)\pi}{8}i}\ \forall\ k=0,1,2,...,7.$ Hence

$Z^8=-1\Rightarrow\left(Z-e^{\frac{\pi}{8}i}\right)\left(Z-e^{\frac{3\pi}{8}i}\right)...\left(Z-e^{\frac{15\pi}{8}i}\right)$, where $\left\{e^{\frac{\pi}{8}i},e^{\frac{15\pi}{8}i}\right\},\left\{e^{\frac{3\pi}{8}i},e^{\frac{13\pi}{8}i}\right\}$...etc. are conjugate pairs.

$$Z^8+1=\left(Z^2-\left(e^{\frac{\pi}{8}i}+e^{\frac{15\pi}{8}i}\right)Z+1\right)\left(Z^2-\left(e^{\frac{3\pi}{8}i}+e^{\frac{13\pi}{8}i}\right)Z+1\right)\left(Z^2-\left(e^{\frac{5\pi}{8}i}+e^{\frac{11\pi}{8}i}\right)Z+1\right)\left(Z^2-\left(e^{\frac{7\pi}{8}i}+e^{\frac{9\pi}{8}i}\right)Z+1\right)$$

$$Z^8+1=\left(Z^2-2\cos\frac{\pi}{8}Z+1\right)\left(Z^2-2\cos\frac{3\pi}{8}Z+1\right)\left(Z^2-2\cos\frac{5\pi}{8}Z+1\right)\left(Z^2-2\cos\frac{7\pi}{8}Z+1\right)$$

Further the above result implies

$$Z^4+\frac{1}{Z^4}=\left(Z+\frac{1}{Z}-2\cos\frac{\pi}{8}\right)\left(Z+\frac{1}{Z}-2\cos\frac{3\pi}{8}\right)\left(Z+\frac{1}{Z}-2\cos\frac{5\pi}{8}\right)\left(Z+\frac{1}{Z}-2\cos\frac{7\pi}{8}\right)$$

Now for $Z=\cos\theta+i\sin\theta$ we get $\cos4\theta=8\left(\cos\theta-\cos\frac{\pi}{8}\right)\left(\cos\theta-\cos\frac{3\pi}{8}\right)\left(\cos\theta-\cos\frac{5\pi}{8}\right)\left(\cos\theta-\cos\frac{7\pi}{8}\right).$

Now for $\theta=0,\left(1-\cos\frac{\pi}{8}\right)\left(1-\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{5\pi}{8}\right)\left(1-\cos\frac{7\pi}{8}\right)=\frac{1}{8}$

or $\left(2\sin^2\frac{\pi}{16}\right)\left(2\sin^2\frac{3\pi}{16}\right)\left(2\sin^2\frac{5\pi}{16}\right)\left(2\sin^2\frac{7\pi}{16}\right)=\frac{1}{8}$

Hence $\sin^2\frac{\pi}{16}\sin^2\frac{3\pi}{16}\sin^2\frac{5\pi}{16}\sin^2\frac{7\pi}{16}=\frac{1}{128}.$

Q.59

(a) $\frac{Z_2-Z_1}{Z_3-Z_1}=\cos\frac{\pi}{3}+i\cos\frac{\pi}{3}\Rightarrow\left(\frac{Z_2-Z_1}{Z_3-Z_1}-\frac{1}{2}\right)^2=-\frac{3}{4}$

or $\left(\frac{Z_2-Z_1}{Z_3-Z_1}\right)^2-2\left(\frac{Z_2-Z_1}{Z_3-Z_1}\right)+1=0$

$\therefore Z_1{}^2+Z_2{}^2+Z_3{}^2=Z_1Z_2+Z_2Z_3+Z_3Z_1.$

(b) $Z_1 + Z_2 + Z_3 = 3Z_0 \Rightarrow Z_1^2 + Z_2^2 + Z_3^2 + 2Z_1Z_2 + 2Z_2Z_3 + 2Z_3Z_1 = 9Z_0^2$

Also for an equilateral Δ, $Z_1^2 + Z_2^2 + Z_3^2 = Z_1Z_2 + Z_2Z_3 + Z_3Z_1$.

Hence $Z_1^2 + Z_2^2 + Z_3^2 = 3Z_0^2$.

(c) $Z_1 + Z_2 + \ldots + Z_{3n} = 3nZ_0$ & $(Z_1 - Z_0)^2 + (Z_2 - Z_0)^2 + \ldots + (Z_{3n} - Z_0)^2 = 0$

$\Rightarrow \left(Z_1^2 + Z_2^2 + \ldots + Z_{3n}^2\right) - 2Z_0\left(Z_1 + Z_2 + \ldots + Z_{3n}\right) + 3nZ_0^2 = 0$

or $Z_1^2 + Z_2^2 + \ldots + Z_{3n}^2 = 3nZ_0^2$

Q.60

(a) Using $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$ we get

$\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{6\pi}{7} = 4\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}$.

Now $\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \frac{\sqrt{7}}{2}$.

(b) Let center be the origin and A_1 be on real-axis i.e. $OA_1 = 1$

Now each of the internal angle will be $\frac{2\pi}{7}$, hence $OA_2 = e^{\frac{2i\pi}{7}}, OA_3 = e^{\frac{4i\pi}{7}}$ & $OA_4 = e^{\frac{6i\pi}{7}}$.

$\Rightarrow A_1A_2 = OA_2 - OA_1 = e^{\frac{2i\pi}{7}} - 1, A_1A_3 = e^{\frac{4i\pi}{7}} - 1$ & $A_1A_4 = OA_4 - OA_1 = e^{\frac{6i\pi}{7}} - 1$.

Hence $|A_1A_2| + |A_1A_4| - |A_1A_3| = \left|e^{\frac{2i\pi}{7}} - 1\right| + \left|e^{\frac{4i\pi}{7}} - 1\right| - \left|e^{\frac{6i\pi}{7}} - 1\right|$.

$\Rightarrow 2\sin\frac{\pi}{7} + 2\sin\frac{2\pi}{7} - 2\sin\frac{3\pi}{7} = \sqrt{7}$.