Complex Numbers

Exercise – 2(C)

Q.1

Let $Z = x + iy$,

then equation of line becomes $x + y = k$ & equation of circle becomes $x^2 + y^2 - 2x - 4y - 13 = 0.$

Now center of the circle is $(1,2)$ & radius $= 3\sqrt{2}.$

As the line is a secant hence $\left|\frac{1+2-k}{\sqrt{2}}\right| < 3\sqrt{2}$ or $|k-3| < 6.$

Hence $-3 < k < 9.$

Q.2

If Z_1, Z_2 & Z_3 form an equilateral triangle, then $Z_1^2 + Z_2^2 + Z_3^2 = Z_1Z_2 + Z_2Z_3 + Z_3Z_1.$

Now $Z_1 = a + i, Z_2 = 1 + bi$ & $Z_3 = 0$

$\Rightarrow a^2 - 1 + 2ai + 1 - b^2 + 2bi = a - b + (1 + ab)i$

Or $a^2 - b^2 = a - b$ & $2a + 2b = 1 + ab.$

Hence $a + b = 1$ or $a = b.$

$(i)\, a + b = 1$ gives $ab = 1$, hence $a^2 - a + 1 = 0$, which doesnt have real roots.

$(ii)\, a = b$ gives $a^2 - 4a + 1 = 0$, hence $a = b = 2 - \sqrt{3}$ (as a & $b < 1$)

Now $a^2 + b^2 = 2\left(2 - \sqrt{3}\right)^2 = 14 - 8\sqrt{3}.$

As $0 < 14 - 8\sqrt{3} < 1$, hence $\left[a^2 + b^2\right] = 0.$

Q.3

$Z = \frac{\sqrt{3} - i}{2}$ or $Z = i\omega^2 \Rightarrow Z^3 = -i$ & $Z^2 = -\omega$, where ω & ω^2 are complex cube roots of unity.

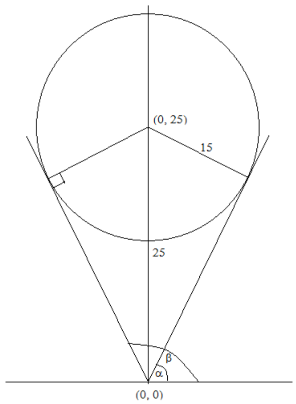
Now $Z^{95} = Z^{3\times 31} \times Z^2 = -i\omega$ & $i^{67} = i^{4\times 16} \times i^3 = -i$

$\left(Z^{95} + i^{67}\right)^{94} = i^{94}(\omega + 1)^{94} = -\omega^{188}$

$\Rightarrow \left(Z^{95} + i^{67}\right)^{94} = -\omega^2.$

Now $Z^n = i^n \omega^{2n}$, hence least possible value of n is 10.

Q.4



$|Z-25i|\le 15 \Rightarrow P(Z)$ lies on a circle of radius 15 & having center at $(0, 25)$.

Let $\text{Max.Arg}(Z)=\alpha$ & $\text{Min.Arg}(Z)=\beta$, then clearly $\tan\alpha$ & $\tan\beta$ will be slopes of tangents to this circle from O.

As shown in figure $\sin(90-\alpha)=\frac{15}{25}$ or $\cos\alpha=\frac{3}{5}\Rightarrow \alpha=\cos^{-1}\frac{3}{5}$.

Similarly $\sin(\beta-90)=\frac{3}{5}$ or $\cos\beta=-\frac{3}{5}\Rightarrow\beta=\pi-\cos^{-1}\frac{3}{5}$.

Now $|\beta-\alpha|=\pi-2\cos^{-1}\frac{3}{5}=2\cos^{-1}\frac{4}{5}$.

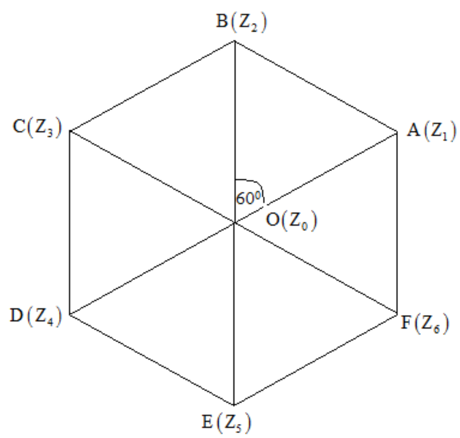
Q.5

Let $Z_i=\cos\theta_i+i\sin\theta_i$ (where $\theta_i=\alpha,\beta,\gamma$), then as given $Z_1+Z_2+Z_3=0$.

$\Rightarrow Z_1^3+Z_2^3+Z_3^3=3Z_1Z_2Z_3$

$\Rightarrow \cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma)$, hence $\lambda=1$.

Q.6



As Z_0, Z_i & Z_{i+1} form an equilateral triangle hence $Z_0^2+Z_i^2+Z_{i+1}^2=Z_0Z_i+Z_0Z_{i+1}+Z_iZ_{i+1}$. Hence

$$\sum_{i=1}^{6}\left(Z_0^2+Z_i^2+Z_{i+1}^2\right)=\sum_{i=1}^{6}\left(Z_0Z_i+Z_0Z_{i+1}+Z_iZ_{i+1}\right),$$

where $Z_7=Z_1$.

$$\Rightarrow 6Z_0^2+2\sum_{i=1}^{6}Z_i^2=2Z_0\sum_{i=1}^{6}Z_i$$

Also $Z_0=\dfrac{\sum_{i=1}^{6}Z_i}{6}\Rightarrow\sum_{i=1}^{6}Z_i^2=3Z_0^2$.

Q.7

$$\sin\frac{2q\pi}{11}-i\cos\frac{2q\pi}{11}=-i\left(\cos\frac{2q\pi}{11}+i\sin\frac{2q\pi}{11}\right).\ \text{But } \cos\frac{2q\pi}{11}+i\sin\frac{2q\pi}{11},\ \text{for } q=0,1,2,\ldots,10$$

gives 11th cube roots of unity, hence $\sum_{q=0}^{10}\left(\cos\frac{2q\pi}{11}+i\sin\frac{2q\pi}{11}\right)=0.$

Now $\sum_{i=1}^{10}\left(\sin\frac{2q\pi}{11}-i\cos\frac{2q\pi}{11}\right)=(-i)\sum_{i=1}^{10}\left(\cos\frac{2q\pi}{11}+i\sin\frac{2q\pi}{11}\right)$

Or $\sum_{i=1}^{10}\left(\sin\frac{2q\pi}{11}-i\cos\frac{2q\pi}{11}\right)=i-i\sum_{i=0}^{10}\left(\cos\frac{2q\pi}{11}+i\sin\frac{2q\pi}{11}\right)^{p}=i.$

Now $\sum_{p=1}^{32}(3p+2)\left(\sum_{i=1}^{10}\left(\sin\frac{2q\pi}{11}-i\cos\frac{2q\pi}{11}\right)\right)^{p}=\sum_{p=1}^{32}(3p+2)i^{p}$

Let $S=5i+8i^2+11i^3+\ldots+98i^{32}$, then

$iS=\quad 5i^2+\ 8i^3+\ldots+95i^{32}+98i^{33}$

$(1-i)S=5i+3i^2+3i^3+\ldots3i^{32}-98i^{33}$

Or $S=\frac{-96i}{1-i}+\frac{3i\left(1-i^{32}\right)}{(1-i)^2}\Rightarrow S=\frac{-96i(1+i)}{2}=48-48i$, hence $a+b=0.$

Q.8

$Z^6+Z^5+Z^4+Z^3+Z^2+Z+1=0\Rightarrow Z^7-1=0, Z\neq 1.$

Hence $Z_i, i=1,2,\ldots,6$ are 7th roots of unity except 1 itself.

Now $\sum_{i=0}^{6}Z_i^{\,r}=\begin{cases}7 & \text{if } r=7k\\ 0 & \text{otherwise}\end{cases}$, hence $\sum_{i=0}^{6}Z_i^{\,5}=0$ & $\sum_{i=0}^{6}Z_i^{\,14}=7.$

$\Rightarrow\sum_{i=1}^{6}Z_i^{\,5}=-1$ & $\sum_{i=1}^{6}Z_i^{\,14}=6.$

Also from the given equation $\prod_{i=1}^{6}Z_i=1.$

Now $\sum_{i=1}^{6}Z_i^{\,5}+\sum_{i=1}^{6}Z_i^{\,14}-\prod_{i=1}^{6}Z_i=-1+6-1=4.$

Q.9

$$\left||z|-\frac{4}{|z|}\right|\le\left|z-\frac{4}{z}\right|\Rightarrow\left||z|-\frac{4}{|z|}\right|\le 3$$

$$\Rightarrow -3\le|z|-\frac{4}{|z|}\le 3$$

$$\Rightarrow |z|^2 - 3|z| - 4 \le 0 \ \& \ |z|^2 + 3|z| - 4 \ge 0$$
$$\Rightarrow 1 \le |z| \le 4$$

Q.10

$$z^3 = 343 \Rightarrow (z-7)(z^2+7z+49) = 0$$
$$\text{Hence } z^2 + 7z + 49 \equiv z^2 + az + b$$
$$\Rightarrow a = 7, b = 49$$
$$\Rightarrow \frac{7a+b}{14} = 7$$

Q.11

$$|z-4| = |z-8| \Rightarrow (z-4)(\bar{z}-4) = (z-8)(\bar{z}-8)$$
$$\Rightarrow z + \bar{z} = 12 \Rightarrow \operatorname{Re}(z) = 6$$
$$\text{Let } z = 6 + ai$$
$$\text{Now } 3|z-12| = 5|z-8i| \Rightarrow 3|ai-6| = 5|6+(a-8)i|$$
$$\Rightarrow 9(a^2+36) = 25(a^2 - 16a + 100)$$
$$\Rightarrow a^2 - 25a + 136 = 0$$
$$\Rightarrow a = 8, 17$$
$$\text{Hence } \operatorname{Im}(z) = 8$$

Q.12

$$\text{Let } a = a_1 + a_2 i \ \& \ b = b_1 + b_2 i$$
$$f(z) = (4+i)z^2 + (a_1 + a_2 i)z + b_1 + b_2 i$$
$$f(1) = a_1 + a_2 i + b_1 + b_2 i + (4+i)$$
$$f(i) = a_1 i - a_2 + b_1 + b_2 i - (4+i)$$
$$f(1) = \overline{f(1)} \Rightarrow a_1 + a_2 i + b_1 + b_2 i + (4+i) = a_1 - a_2 i + b_1 - b_2 i + (4-i)$$
$$\Rightarrow a_2 + b_2 = -1 \ \ldots\text{(i)}$$
$$f(i) = \overline{f(i)} \Rightarrow a_1 i - a_2 + b_1 + b_2 i - (4+i) = -a_1 i - a_2 + b_1 - b_2 i - (4-i)$$
$$\Rightarrow a_1 + b_2 = 1 \ \ldots\text{(ii)}$$
$$\Rightarrow a_1^2 + a_2^2 = 2 + 2b_2^2$$
$$\Rightarrow |a| + |b| = \sqrt{2(1+b_2^2)} + \sqrt{b_1^2 + b_2^2}$$
$$\Rightarrow |a| + |b| \ge \sqrt{2}$$

Q.13

$$z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 = 0 \quad ...(i)$$

$$\Rightarrow \bar{z}^4 + a_1 \bar{z}^3 + a_2 \bar{z}^2 + a_3 \bar{z} + a_4 = 0$$

If z is purely imaginary, then $\bar{z} = -z$

$$\Rightarrow z^4 - a_1 z^3 + a_2 z^2 - a_3 z + a_4 = 0 \quad ...(ii)$$

From (i) & (ii)

$$z^4 + a_2 z^2 + a_4 = 0 \ \& \ a_1 z^2 + a_3 = 0$$

$$\Rightarrow \left(\frac{a_3}{a_1}\right)^2 - a_2\left(\frac{a_3}{a_1}\right) + a_4 = 0$$

$$\Rightarrow \frac{a_3}{a_1 a_2} + \frac{a_1 a_4}{a_2 a_3} = 1$$

Q.14

$$z^3 + \left(\overline{w}\right)^7 = 0 \ \& \ z^5 w^{11} = 1 \Rightarrow |z| = |w| = 1$$

Now $z^3 + \left(\overline{w}\right)^7 = 0 \Rightarrow z^3 w^7 = -1$

Further $z^3 w^7 = -1$ & $z^5 w^{11} = 1$

$$\Rightarrow z = w = \mp i$$

Hence $(z, w) \equiv (i, i)$ or $(-i, -i)$

Q.15

$$\frac{z_1^2}{z_2 z_3} + \frac{z_2^2}{z_1 z_3} + \frac{z_3^2}{z_1 z_2} + 1 = 0 \Rightarrow z_1^3 + z_2^3 + z_3^3 - 3z_1 z_2 z_3 = -4z_1 z_2 z_3$$

$$\Rightarrow (z_1 + z_2 + z_3)\left(z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1\right) = -4z_1 z_2 z_3$$

$$\Rightarrow (z_1 + z_2 + z_3)^3 - 3(z_1 + z_2 + z_3)(z_1 z_2 + z_2 z_3 + z_3 z_1) + 4z_1 z_2 z_3 = 0$$

Let $z = z_1 + z_2 + z_3$, then

$$z^3 - 3z(z_1 z_2 + z_2 z_3 + z_3 z_1) + 4z_1 z_2 z_3 = 0$$

$$\Rightarrow z^3 = z_1 z_2 z_3 \left\{3z\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) - 4\right\}$$

$$\Rightarrow z^3 = z_1 z_2 z_3 \left\{3z\left(\bar{z}_1 + \bar{z}_2 + \bar{z}_3\right) - 4\right\}$$

$$\Rightarrow |z|^3 = 3|z|^2 - 4$$

$$\Rightarrow |z| = 1, 2$$