

## COMPLEX NUMBERS

### Exercise – 2(A)

**Q.1 (a)**

$$|Z_1| = |Z_2| \Rightarrow Z_1 \bar{Z}_1 = Z_2 \bar{Z}_2 \text{ or } \frac{Z_1}{Z_2} = \frac{\bar{Z}_2}{\bar{Z}_1}$$

$$\Rightarrow \frac{Z_1 + Z_2}{Z_1 - Z_2} = -\frac{\bar{Z}_1 + \bar{Z}_2}{\bar{Z}_1 - \bar{Z}_2}$$

Hence  $\frac{Z_1 + Z_2}{Z_1 - Z_2}$  is purely imaginary.

**Q.2 (b)**

$$|Z|^{n-2} Z^2 + |Z|^{n-2} Z - |Z|^n = 0 \Rightarrow |Z|^{n-2} (Z^2 + Z - |Z|^2) = 0,$$

$$\text{hence } |Z| = 0 \text{ or } Z^2 + Z - |Z|^2 = 0$$

$$\Rightarrow x - 2y^2 + y(2x+1)i = 0.$$

$$\text{Hence } x = 2y^2 \text{ \& } y = 0 \left\{ \text{given } x \neq -\frac{1}{2} \right\}.$$

Therefore  $x = 0, y = 0$ .

Hence one solution is possible as  $Z = 0$  if  $n \geq 2$ .

**Q.3 (b)**

Let the root be  $yi$ , then  $Z^4 + a_1 Z^3 + a_2 Z^2 + a_3 Z + a_4 = 0$  gives

$$y^4 - a_1 y^3 - a_2 y^2 + a_3 y + a_4 = 0 \dots (i)$$

Also as the coefficients are real, hence  $-yi$  must also be a root.

$$\therefore y^4 + a_1 y^3 - a_2 y^2 - a_3 y + a_4 = 0 \dots (ii)$$

$$\text{Subtracting (i) from (ii) we get } y^2 = \frac{a_3}{a_1}$$

$$\text{Adding (i) \& (ii) we get } y^4 - a_2 y^2 + a_4 = 0$$

$$\therefore \frac{a_3^2}{a_1^2} - \frac{a_2 a_3}{a_1} + a_4 = 0 \text{ or } \frac{a_3}{a_1 a_2} + \frac{a_1 a_4}{a_2 a_3} = 1.$$

**Q.4 (b)**

$$(A+1)^n = A^n \Rightarrow A+1 = A e^{\frac{2k\pi}{n}i} \therefore A = \frac{1}{e^{\frac{2k\pi}{n}i} - 1} \text{ or } \frac{\cos \frac{2k\pi}{n} - i \sin \frac{2k\pi}{n} - 1}{2 - 2 \cos \frac{2k\pi}{n}}$$

$$\Rightarrow A = i \frac{\cos \frac{k\pi}{n} - i \sin \frac{k\pi}{n}}{2 \sin \frac{k\pi}{n}} \text{ or } A^n = i^n \frac{\cos k\pi}{2^n \sin^n \frac{k\pi}{n}}$$

Now  $i^n \cos k\pi = 2^n \sin^n \frac{k\pi}{n}$ , therefore least value of  $n = 6$ .

### Q.5 (c)

$$\text{Given } Z_1(Z_1^2 - 3Z_2^2) = 2 \text{ \& } Z_2(3Z_1^2 - Z_2^2) = 11.$$

$$\text{Now } Z_1(Z_1^2 - 3Z_2^2) + i Z_2(3Z_1^2 - Z_2^2) = 2 + 11i \Rightarrow (Z_1 + iZ_2)^3 = 2 + 11i \dots (i)$$

$$\text{\& } Z_1(Z_1^2 - 3Z_2^2) - i Z_2(3Z_1^2 - Z_2^2) = 2 - 11i \Rightarrow (Z_1 - iZ_2)^3 = 2 - 11i \dots (ii)$$

$$\text{Further } (Z_1 + iZ_2)^3 (Z_1 - iZ_2)^3 = 125 \text{ or } Z_1^2 + Z_2^2 = 5.$$

### Q.6 (b)

$$\text{Let } Z = x + iy, \text{ then } |2Z - 1| = |Z - 2| \Rightarrow (2x - 1)^2 + 4y^2 = (x - 2)^2 + y^2$$

$$\text{or } x^2 + y^2 = 1.$$

$$\text{Now } |(Z_1 - \alpha) + (Z_2 - \beta) + \alpha + \beta| \leq |Z_1 - \alpha| + |Z_2 - \beta| + |\alpha| + |\beta|$$

$$\Rightarrow |Z_1 + Z_2| \leq 2(\alpha + \beta) \text{ or } \left| \frac{Z_1 + Z_2}{\alpha + \beta} \right| \leq 2$$

$$\text{Hence } \left| \frac{Z_1 + Z_2}{\alpha + \beta} \right| \leq 2|Z|.$$

### Q.7 (c)

$$\begin{aligned}
& |a_0 Z^n + a_1 Z^{n-1} + a_2 Z^{n-2} + \dots + a_n| = 3 \\
& \Rightarrow |a_0| |Z|^n + |a_1| |Z|^{n-1} + |a_2| |Z|^{n-2} + \dots + |a_n| \geq 3 \\
& \Rightarrow |Z|^n + |Z|^{n-1} + |Z|^{n-2} + \dots + 1 > \frac{3}{2} \text{ as } |a_i| < 2
\end{aligned}$$

when  $|Z| \geq 1$ , anyway the inequality holds so when  $|Z| < 1$ , then LHS will be greatest as  $n \rightarrow \infty$ .

$$\text{Hence } \frac{1}{1-|Z|} > \frac{3}{2} \text{ or } |Z| > \frac{1}{3}.$$

**Q.8 (b)**

$$|Z| = \left| Z + \frac{1}{Z} - \frac{1}{Z} \right| \Rightarrow |Z| \leq \left| Z + \frac{1}{Z} \right| + \frac{1}{|Z|}$$

$$\therefore \left| Z + \frac{1}{Z} \right| \geq |Z| - \frac{1}{|Z|} \text{ So } \left| Z + \frac{1}{Z} \right| \geq \frac{8}{3}.$$

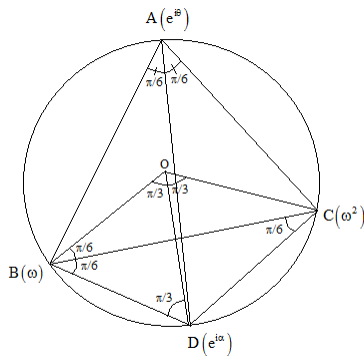
**Q.9 (c)**

$$|Z_r - r| \geq ||Z_r| - r| \Rightarrow ||Z_r| - r| \leq r \text{ or } |Z_r| \leq 2r.$$

$$\text{Hence } \sum_{r=1}^n |Z_r| \leq n(n+1)$$

$$\text{Also } \left| \sum_{r=1}^n Z_r \right| \leq \sum_{r=1}^n |Z_r|, \text{ therefore } \left| \sum_{r=1}^n Z_r \right| \leq n(n+1)$$

**Q.10 (d)**



Refer the adjoining figure.

$$\frac{1}{2} \angle BOC = \angle BAC = \frac{\pi}{3} \text{ \& } \angle CAD = \angle BAD = \angle CBD = \frac{\pi}{6}$$

$$\Rightarrow \angle BOD = \frac{\pi}{3}$$

Hence D is  $(-1, 0)$ .

$$\therefore OD = \omega + \omega^2$$

**Q.11 (c)**

$$|t|=1 \Rightarrow |Zc+b|=|Z-a| \text{ or } \frac{\left|Z+\frac{b}{c}\right|}{|Z-a|}=|c|. \text{ As } |c| \neq 1, \text{ hence locus will be a circle.}$$

$$Z_k = 1 + a + a^2 + \dots + a^{k-1} \Rightarrow Z_k = \frac{1-a^k}{1-a} \text{ or } Z_k - \frac{1}{1-a} = -\frac{a^k}{1-a}$$

$$\Rightarrow \left| Z_k - \frac{1}{1-a} \right| = \left| \frac{a^k}{1-a} \right| < \frac{1}{|1-a|}. \text{ Hence } Z_k \text{ lies inside the circle } \left| Z - \frac{1}{1-a} \right| = \frac{1}{|1-a|}.$$

$$Z = \frac{1}{2 + \cos \theta + i \sin \theta} \Rightarrow Z = \frac{1}{2 + e^{i\theta}} \text{ or } e^{i\theta} = \frac{1 - 2Z}{Z}.$$

$$\text{or } 3x^2 + 3y^2 - 4x + 1 = 0.$$

Hence locus of Z is a circle with center at x – axis.

**Q.15 (d)**

$p^{\text{th}}$  roots of unity  $= e^{\frac{2k_1\pi}{p}i}$  for  $k_1 = 1, 2, 3, \dots, p-1$  &

$q^{\text{th}}$  roots of unity  $= e^{\frac{2k_2\pi}{q}i}$  for  $k_2 = 1, 2, 3, \dots, q-1$

Now let  $\frac{k_1}{p} = \frac{k_2}{q}$  for some  $\{k_1, k_2\}$ , then  $qk_1 = pk_2$ .

As  $p$  &  $q$  are distinct prime numbers hence the above conclusion is a contradiction.

Number of such  $\{p, q\} = 0$ .

**Q.16 (d)**

$$\left(\frac{Z+1}{Z}\right)^n = 1 \Rightarrow \frac{Z+1}{Z} = e^{\frac{2k\pi}{n}i} \text{ for } k = 1, 2, \dots, n-1.$$

$$\text{or } Z = \frac{1}{\cos \frac{2k\pi}{n} - 1 + i \sin \frac{2k\pi}{n}} \text{ or } Z = \frac{\cos \frac{2k\pi}{n} - 1 - i \sin \frac{2k\pi}{n}}{\left(\cos \frac{2k\pi}{n} - 1\right)^2 + \sin^2 \frac{2k\pi}{n}}$$

$$\Rightarrow \text{Re}(Z) = \frac{\cos \frac{2k\pi}{n} - 1}{2 - 2 \cos \frac{2k\pi}{n}} \text{ or } \text{Re}(Z) = -\frac{1}{2}$$

$$\text{Hence } \sum_{k=1}^{n-1} \text{Re}(Z) = -\frac{n-1}{2}.$$

**Q.17 (c)**

$$\left(\frac{Z+1}{Z}\right)^4 = 16 \Rightarrow \frac{Z+1}{Z} = -2, 2, -2i, 2i \text{ or } Z = -\frac{1}{3}, 1, -\frac{1-2i}{5} \& -\frac{1+2i}{5}.$$

Points representing these roots on argand plane are

$$A\left(-\frac{1}{3}, 0\right), B(1, 0), C\left(-\frac{1}{5}, \frac{2}{5}\right) \& D\left(-\frac{1}{5}, -\frac{2}{5}\right).$$

Point which is equidistant from these is  $\left(\frac{1}{3}, 0\right)$ .

**Q.18 (a)**

$$(Z-1)(Z-Z_1)(Z-Z_2)\dots(Z-Z_{n-1})=Z^n-1\Rightarrow(Z-Z_1)(Z-Z_2)\dots(Z-Z_{n-1})=\frac{Z^n-1}{Z-1}$$

$$\text{or } \ln(Z-Z_1)+\ln(Z-Z_2)+\dots+\ln(Z-Z_{n-1})=\ln(Z^n-1)+\ln(Z-1)$$

Differentiating w.r.to Z we get

$$\frac{1}{Z-Z_1}+\frac{1}{Z-Z_2}+\dots+\frac{1}{Z-Z_{n-1}}=\frac{nZ^{n-1}}{Z^n-1}+\frac{1}{Z-1}.$$

Substitute  $Z=3$  gives

$$\frac{1}{3-Z_1}+\frac{1}{3-Z_2}+\dots+\frac{1}{3-Z_{n-1}}=\frac{n3^{n-1}}{3^n-1}+\frac{1}{2}.$$

**Q.19 [d]**

$$z^3-2z^2+z-1=0 \quad \dots\dots\dots \alpha, \beta, \gamma$$

$$\text{Let } y = \frac{1}{z-1}$$

$$\Rightarrow z = \frac{1+y}{y}$$

By theory of equations,

$$\Rightarrow \left(\frac{1+y}{y}\right)^3 - 2\left(\frac{1+y}{y}\right)^2 + \left(\frac{1+y}{y}\right) - 1 = 0 \text{ will have the roots } \frac{1}{\alpha-1}, \frac{1}{\beta-1} \& \frac{1}{\gamma-1}$$

$$\Rightarrow y^3 - y - 1 = 0 \quad \dots\dots\dots \frac{1}{\alpha-1}, \frac{1}{\beta-1}, \frac{1}{\gamma-1}$$

$$\text{Sum of roots } \frac{1}{\alpha-1} + \frac{1}{\beta-1} + \frac{1}{\gamma-1} = 0$$

$$\text{Required value} = \frac{\alpha}{\alpha-1} + \frac{\beta}{\beta-1} + \frac{\gamma}{\gamma-1}$$

$$\Rightarrow \frac{\alpha-1+1}{\alpha} + \frac{\beta-1+1}{\beta-1} + \frac{\gamma-1+1}{\gamma-1}$$

$$\Rightarrow 3 + \left( \frac{1}{\alpha-1} + \frac{1}{\beta-1} + \frac{1}{\gamma-1} \right)$$

$$\Rightarrow 3 + 0 = 3.$$

**Q.20 [b]**

Point Q( $z_2$ ) will be image of P ( $1 + i$ ) in the line  $(3 - 4i)z + (3 + 4i)\bar{z} + 1 = 0$ , if

$$\Rightarrow (3 - 4i)z_2 + (3 + 4i)(\overline{1 + i}) + 1 = 0$$

$$\Rightarrow z_2 = \frac{-[(3 + 4i)(1 - i) + 1]}{(3 - 4i)} = -\left(\frac{4 + 7i}{5}\right)$$

**Q.21 [a]**

Let  $z = ib$

$$\Rightarrow |z - 2 - 3i|^2 + |z + 4 - i|^2 = 4|z - 1 + 2i|^2$$

$$\Rightarrow (a - 2)^2 + (b - 3)^2 + (a + 4)^2 + (b - 1)^2 = 4[(a - 1)^2 + (b + 2)^2]$$

$$\Rightarrow a^2 + b^2 - 6a + 12b - 5 = 0$$

$$\Rightarrow \text{radius of circle} = \sqrt{3^2 + 6^2 + 5} = 5\sqrt{2}$$

**Q.22 [d]**

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ 2 & 1 + \omega & 1 + \omega^2 \\ 3 & 2 + \omega & 2 + \omega^2 \end{vmatrix}$$

$$C_3 \rightarrow C_3 - C_2$$

$$\Rightarrow \begin{vmatrix} 1 & \omega & \omega^2 - \omega \\ 2 & 1 + \omega & \omega^2 - \omega \\ 3 & 2 + \omega & \omega^2 - \omega \end{vmatrix}$$

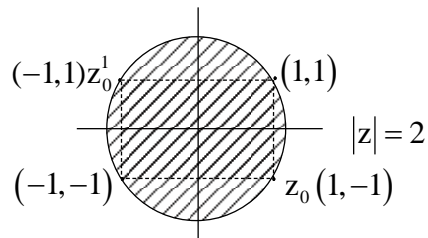
$$\Rightarrow (\omega^2 - \omega) \begin{vmatrix} 1 & \omega & 1 \\ 2 & 1 + \omega & 1 \\ 3 & 2 + \omega & 1 \end{vmatrix}$$

$$\Rightarrow C_2 \rightarrow C_2 + C_3 - C_1$$

$$\Rightarrow \begin{vmatrix} 1 & \omega & 1 \\ 2 & \omega & 1 \\ 3 & \omega & 1 \end{vmatrix}$$

$$\Rightarrow 0$$

**Q.23 [d]**



By geometry, one can observe that greatest value of  $|z - (1+i)| + |z - (-1-i)|$  will occur if  $z = z_0$  or  $z = z_0^1$ .

Hence, maximum value will be

$$2 + 2 = 4$$

**Q.24 [b]**

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \dots\dots\dots(i)$$

$$\Rightarrow e^{-i\theta} = \cos \theta - i \sin \theta \quad \dots\dots\dots(ii)$$

From (i) & (ii)

$$\Rightarrow \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\Rightarrow \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\Rightarrow \tan \theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$$

$$\Rightarrow \tan(i \sin \theta) = \frac{(e^{-\sin \theta} - e^{\sin \theta})}{i(e^{-\sin \theta} + e^{\sin \theta})}$$

Hence, purely imaginary



**Q.25 [a]**

From Binomial Theorem, we know that

$$\Rightarrow (1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_nx^n$$

Substituting  $x = 1$ ,

$$\Rightarrow 2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots \quad \dots\dots(i)$$

Substituting  $x = -1$

$$\Rightarrow 0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + {}^nC_4 - \dots \quad \dots\dots(ii)$$

Adding (i) & (ii)

$$\Rightarrow 2^{n-1} = ({}^nC_0 + {}^nC_2 + {}^nC_4 + {}^nC_6 + \dots) \quad \dots\dots(iii)$$

$$\text{Also, } (1+ix)^n = {}^nC_0 + {}^nC_1(ix) - {}^nC_2x^2 - {}^nC_3ix^3 + {}^nC_4x^4 + {}^nC_5ix^5 + \dots$$

Substituting  $x = 1$

$$\Rightarrow (1+i)^n = ({}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + \dots) + i({}^nC_1 - {}^nC_3 + {}^nC_5 - {}^nC_7 + \dots)$$

$$\Rightarrow (\sqrt{2})^n e^{i\frac{2\pi}{4}}$$

$$\Rightarrow 2^{\frac{n}{2}} \cos \frac{n\pi}{2} + i 2^{\frac{n}{2}} \sin \frac{4\pi}{4} = ({}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + \dots) + i({}^nC_1 - {}^nC_3 + {}^nC_5 - \dots)$$

Equating real parts of both sides,

$$\Rightarrow 2^{\frac{n}{2}} \cos \frac{n\pi}{4} = {}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 \quad \dots\dots(iv)$$

Adding (iii) & (iv), we get

$$\Rightarrow 2^{n-1} + 2^{\frac{n}{2}} \cos \frac{n\pi}{4} = 2({}^nC_0 + {}^nC_4 + {}^nC_8 + {}^nC_{12} + \dots)$$

Substituting  $n = 20$ ,

$$\Rightarrow \frac{1}{2} (2^{19} + 2^{10} \cos 5\pi) = {}^{20}C_0 + {}^{20}C_4 + {}^{20}C_8 + {}^{20}C_{12} + {}^{20}C_{16} + {}^{20}C_{20}$$

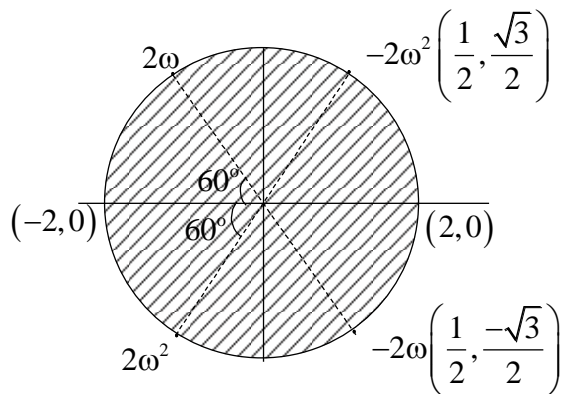
$$\Rightarrow 2^9(2^9 - 1)$$

**Q.26**

$$z^2 - 2z + 4 = 0$$

$$\Rightarrow z = \frac{2 \pm \sqrt{4 - 16}}{2}$$

$$\Rightarrow 1 + \sqrt{3}i \text{ or } 1 - \sqrt{3}i$$



Required value =

$$\left(\frac{-2\omega}{2} + \frac{2}{-2\omega}\right)^2 + \left(\omega^2 + \frac{1}{\omega^2}\right)^2 + \left((- \omega)^3 + (- \omega^2)^3\right)^2 + \left(\omega^4 + \frac{1}{\omega^4}\right)^2 + \left((- \omega)^5 + \frac{1}{(- \omega)^5}\right)^2 + \left((- \omega)^6 + \frac{1}{(- \omega)^6}\right)^2 + \dots$$

$$\Rightarrow (-\omega - \omega^2)^2 + (\omega^2 + \omega)^2 + (1 + 1)^2 + (\omega + \omega^2)^2 + (-\omega - \omega^2)^2 + (1 + 1)^2 + \dots$$

$$\Rightarrow (1 + 1 + 4) + (1 + 1 + 4) + \dots$$

$$\Rightarrow 6 \times 8 = 48$$

$\Rightarrow$  8 times repetition

**Q.27 [b]**

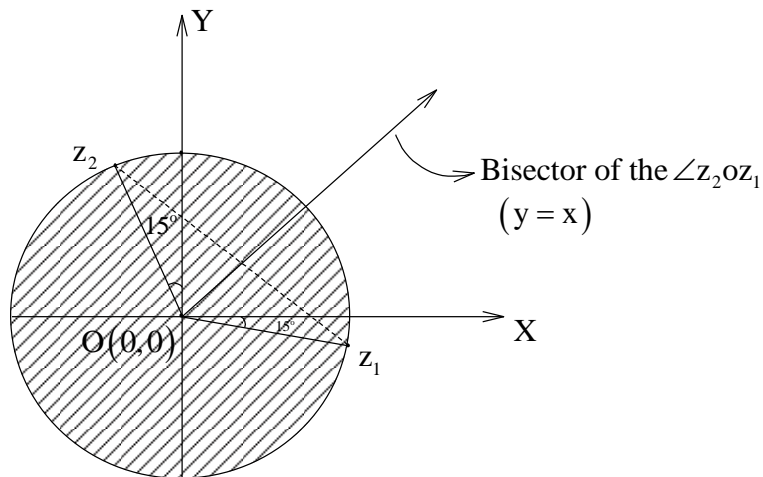
$$z_1 = \left(\frac{\sqrt{3} + 1}{2\sqrt{2}}\right) - \left(\frac{\sqrt{3} - 1}{2\sqrt{2}}\right)i$$

$$\Rightarrow \cos(15^\circ) - i \sin(15^\circ)$$

$$\Rightarrow \cos(-15^\circ) + i \sin(-15^\circ)$$

$$\Rightarrow z_2 = -\sin(15^\circ) + i \cos(15^\circ)$$

$$\Rightarrow \cos(90^\circ + 15^\circ) + i \sin(90^\circ + 15^\circ)$$



**Q.28 [b]**

$$z_n = e^{i \frac{\pi}{(2n+1)(2n+3)}}$$

$$\Rightarrow z_1 z_2 z_3 \dots z_n = e^{i \sum_{r=1}^n \frac{\pi}{2} \left( \frac{1}{2r+1} - \frac{1}{2r+3} \right)}$$

$$\Rightarrow e^{i \frac{\pi}{2} \left( \frac{1}{3} - \frac{1}{2n+2} \right)}$$

$$\Rightarrow \lim_{n \rightarrow \infty} (z_1 z_2 z_3 \dots z_n)$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left( e^{i \frac{\pi}{2} \left( \frac{1}{3} - \frac{1}{2n+2} \right)} \right) e^{i \frac{\pi}{6}} = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6}$$

**Q.29 [a]**

$$\Delta OAP \sim \Delta OQR$$

$$\Rightarrow \frac{OR}{OQ} = \frac{OP}{OA}$$

Using concept of Rotation

$$\Rightarrow \frac{Z-0}{|Z-0|} = \frac{Z_2-0}{|Z_2-0|} e^{i\theta} \quad \dots\dots\dots(i)$$

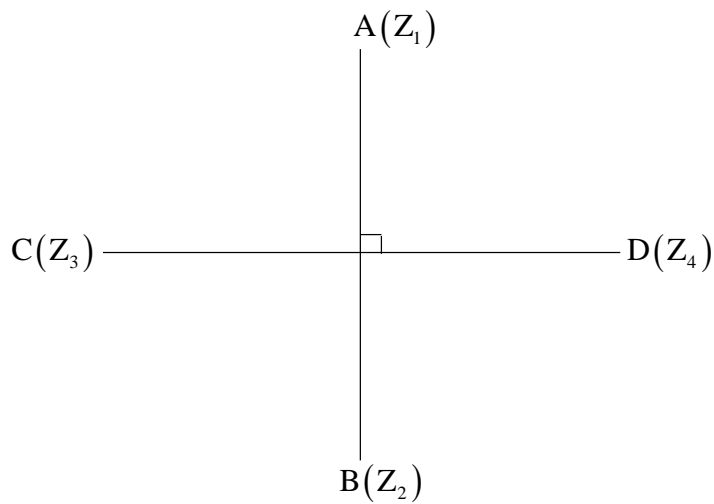
$$\Rightarrow \frac{Z_1-0}{|Z_1-0|} = \frac{1-0}{OA} e^{i\theta} \quad \dots\dots\dots(ii)$$

From eq<sup>n</sup> (i) & (ii)

$$\Rightarrow \frac{Z}{OA \times OR} \times e^{i\theta} = \frac{Z_2 Z_2 e^{i\theta}}{OP \times OQ}$$

$$\Rightarrow Z = Z_1 Z_2$$

**Q.30 [a]**



We know that, if  $AB \perp CB$ , then

$$\Rightarrow \frac{Z_1 - Z_2}{Z_3 - Z_4} + \frac{\overline{Z_1} - \overline{Z_2}}{\overline{Z_3} - \overline{Z_4}} = 0$$

$$\Rightarrow \frac{Z_1 - Z_2}{\overline{Z_1} - \overline{Z_2}} + \frac{Z_3 - Z_4}{\overline{Z_3} - \overline{Z_4}} = 0$$

$$\Rightarrow \omega_1 + \omega_2 = 0$$