COMPLEX NUMBERS

Ex. 1(C)

Q.1 (c)

$x = 9^{\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots\infty \text{ terms}} \Rightarrow x = 9^{\frac{1/3}{1-1/3}}$ or $x = 3.$

$y = 4^{\frac{1}{3}-\frac{1}{9}+\frac{1}{27}-\ldots\infty \text{ terms}} \Rightarrow y = 4^{\frac{1/3}{1+1/3}}$ or $y = \sqrt{2}.$

$z = \sum_{r=1}^{\infty} \frac{1}{(1+i)^r} \Rightarrow z = \frac{1/(1+i)}{1-1/(1+i)}$ or $z = -i.$

Now $x + yz = 3 - \sqrt{2}i \Rightarrow \arg(x + yz) = -\tan^{-1}\frac{\sqrt{2}}{3}.$

Q.2 (c)

$\overline{Z} + i\overline{W} = 0 \Rightarrow Z - iW = 0$ or $\frac{Z}{W} = i.$

Now $\arg\left(\frac{Z}{W}\right) = \frac{\pi}{2} \Rightarrow \arg(ZW) + \arg\left(\frac{Z}{W}\right) = \frac{3\pi}{2}$

$\Rightarrow \arg(Z^2) = \frac{3\pi}{2}$ or $\arg(Z) = \frac{3\pi}{4}.$

Q.3 (d)

Let $P(x) = x^6 + 4x^5 + 3x^4 + 2x^3 + x + 1$

or $P(x) = x^6 - x^3 + (4x^3 - x^2 + 1)(x^2 + x + 1).$

Now for $x = \omega$ & ω^2, $x^6 - x^3 = x^2 + x + 1 = 0$, hence $P(\omega) = 0 = P(\omega^2).$

$P(x)$ is divisible be $(x - \omega)(x - \omega^2).$

Q.4 (c)

$\cos r\theta + i\sin r\theta = e^{ir\theta} \Rightarrow (\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)\ldots(\cos n\theta + i\sin n\theta) = \sum_{r=1}^{n} e^{ir\theta}$

Hence $e^{i\frac{n(n+1)}{2}\theta} = 1$ or $\cos\frac{n(n+1)}{2}\theta + i\sin\frac{n(n+1)}{2}\theta = 1.$

$\Rightarrow \frac{n(n+1)}{2}\theta = 2m\pi$ or $\theta = \frac{4m\pi}{n(n+1)}.$

Q.5 (c)

Let $Z = \cos\theta + i\sin\theta$.

$$\left|\frac{Z}{\bar{Z}} + \frac{\bar{Z}}{Z}\right| = 1 \Rightarrow \left|Z^2 + \bar{Z}^2\right| = 1 \text{ or } 2\left|\cos 2\theta\right| = 1.$$

Now $\cos 2\theta = \pm\frac{1}{2} \Rightarrow \theta = \frac{n\pi}{2} \pm \frac{\pi}{6}$.

As $\theta \in (0, 2\pi)$, therefore $\theta = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6}$.

Q.6 (A)

$\mu^2 - 2\mu + 2 = 0 \Rightarrow (\mu - 1)^2 = -1$ or $\mu - 1 = \pm i$, hence $\alpha - 1 = i$ & $\beta - 1 = -i$.

Now $(x + \mu)^n = (\cot\theta + \mu - 1)^n$

$$\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \frac{(\cot\theta + i)^n - (\cot\theta - i)^n}{2i}$$

$$\Rightarrow \frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \frac{(\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n}{2i\sin^n\theta}$$

or $\dfrac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \dfrac{\sin n\theta}{\sin^n\theta}$.

Q.7 (d)

$|Z - 4| = \operatorname{Re}(Z) \Rightarrow (x-4)^2 + y^2 = x^2$ or $y^2 = 8(x-2)$.

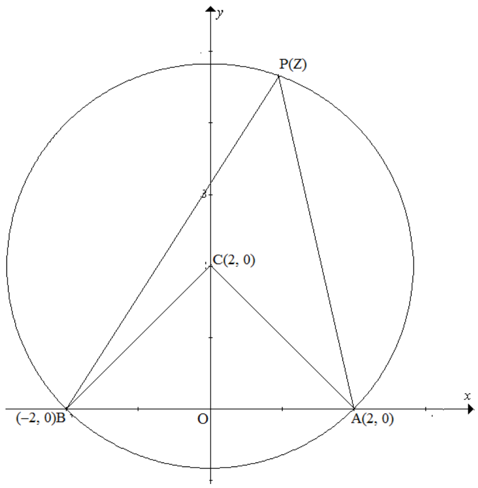
Now greatest positive $\arg(Z)$ will be greatest slope angle of tangent from origin to this parabola.

Equation of any tangent of slope m will be $y = m(x-2) + \frac{2}{m}$.

As it has to be drawn from $(0,0)$, hence $m = 1$.

$\therefore$ Greatest positive $\arg(Z) = \frac{\pi}{4}$.

Q.8 (b)



As shown in figure

both A(2, 0) & B(−2, 0) lie on the circle

$|Z-2i| = 2\sqrt{2}$.

Center of the circle is C(0, 2) & radius is $2\sqrt{2}$.

Now $\angle APB = \angle ACO$.

Hence $\arg\left(\frac{Z-2}{Z+2}\right) = \frac{\pi}{4}$.

Q.9 (b)

$S = 1 + 2\alpha + 3\alpha^2 + 4\alpha^3 + \ldots + n\alpha^{n-1}$ ...(i)

Multiply throughout by α to get

$\alpha S = \alpha + 2\alpha^2 + 2\alpha^3 + \ldots (n-1)\alpha^{n-1} + n\alpha^n$ ...(ii)

Subtract (ii) from (i) to get

$(1-\alpha)S = 1 + \alpha + \alpha^2 + \alpha^3 + \ldots + \alpha^{n-1} - n\alpha^n$

Now $1 + \alpha + \alpha^2 + \alpha^3 + \ldots + \alpha^{n-1} = 0 \Rightarrow S = \frac{-n}{1-\alpha}$.

Q.10 (c)

Let $Z = a + ib$ & $\frac{2}{Z} = x + iy$, then $\frac{2(a-ib)}{a^2+b^2} = x + iy$.

As $a^2 + b^2 = 1$, thus $x = 2a$ & $y = -2b$ or $x^2 + y^2 = 4$.

Required locus is a circle of radius 2.

Q.11 (a)

$\angle AOB = \frac{\pi}{2}$ & $OA = OB \Rightarrow Z_2 = iZ_1$. Hence $\frac{Z_2}{Z_1}$ is purely imaginary.

Q.12 (c)

$\beta+\gamma=\alpha+\alpha^2+...+\alpha^6=-1$ &

$$\beta\times\gamma=\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\right)\left(\cos\frac{6\pi}{7}+\cos\frac{10\pi}{7}+\cos\frac{12\pi}{7}\right)=2$$

hence required equation is $Z^2+Z+2=0.$

Q.13 (c)

Let $Z_1=e^{i\alpha}, Z_2=e^{i\beta}$ & $Z_3=e^{i\gamma}$.

Now $\cos\alpha+2\cos\beta+3\cos\gamma=\sin\alpha+2\sin\beta+3\sin\gamma=0\Rightarrow Z_1+2Z_2+3Z_3=0$

$\Rightarrow Z_1{}^3+8Z_2{}^3+27Z_3{}^3=18Z_1Z_2Z_3$

$\therefore \sin 3\alpha+8\sin 3\beta+27\sin 3\gamma=18\sin(\alpha+\beta+\gamma).$

Q.14 (A)

Let $Z=k(\cos A+i\sin A)$ & $W=k(\cos B+i\sin B)$

Now $\alpha=\dfrac{Z-\overline{W}}{k^2+Z\overline{W}}\Rightarrow\alpha=\dfrac{(\cos A-\cos B)+i(\sin A+\sin B)}{k+k(\cos A+i\sin A)(\cos B-i\sin B)}$

or $\alpha=\dfrac{(\cos A-\cos B)+i(\sin A+\sin B)}{k\{1+\cos(A-B)+i\sin(A-B)\}}$

$$\Rightarrow\alpha=\frac{\sin\frac{A+B}{2}\left(\sin\frac{B-A}{2}+i\cos\frac{A-B}{2}\right)}{k\cos\frac{A-B}{2}\left(\cos\frac{A-B}{2}+i\sin\frac{A-B}{2}\right)}$$

$$\Rightarrow\alpha=\frac{\sin\frac{A+B}{2}\left(-\sin\frac{A-B}{2}+i\cos\frac{A-B}{2}\right)}{k\cos\frac{A-B}{2}\left(\cos\frac{A-B}{2}+i\sin\frac{A-B}{2}\right)}\times\frac{\cos\frac{A-B}{2}-i\sin\frac{A-B}{2}}{\cos\frac{A-B}{2}-i\sin\frac{A-B}{2}}$$

$$\Rightarrow\alpha=\frac{i\sin\frac{A+B}{2}}{k\cos\frac{A-B}{2}}.$$

Hence Re(Z) = 0.

Q.15 (c)

$|Z^2+k|+k=|Z^2|\Rightarrow|Z^2+k|+k=|Z^2+k-k|$

Hence $\arg(Z^2)=-\arg(k)$ or $\arg(Z^2)=\pi.$

$\therefore\ \arg(Z)=\dfrac{\pi}{2}.$

Q.16 (b)

Let $f(Z)=\left(Z^2+1\right)Q(Z)+aZ+b,$

where $Q(Z)$ is the qoutient when $f(Z)$ is divided by Z^2+1

Now $f(i)=i$ & $f(-i)=1+i$, hence

$ai+b=i$ & $-ai+b=1+i.$

Solving these equations simultaneously gives

$b=\frac{1+2i}{2}$ & $a=\frac{i}{2}.$

$\therefore$ remainder when $f(Z)$ is divided by Z^2+1 is $\frac{1+2i}{2}+\frac{iZ}{2}.$

Q.17 (a)

$a|Z_1|=b|Z_2|\Rightarrow\left|\frac{Z_1}{Z_2}\right|=\frac{b}{a}$ or $\frac{a\,Z_1}{b\,Z_2}=e^{i\theta}$ & $\frac{a\,Z_2}{b\,Z_1}=e^{-i\theta}.$

Hence $\frac{a\,Z_1}{b\,Z_2}+\frac{a\,Z_2}{b\,Z_1}=2\cos\theta.$

$\left(\frac{a\,Z_1}{b\,Z_2},\frac{a\,Z_2}{b\,Z_1}\right)$ lies on real axis between $(-2,0)\&(2,0).$

Q.18 (c)

For n^{th} roots of unity

$1+\omega+\omega^2+\omega^3+\ldots+\omega^{n-1}=0$

Also let $S=1+2\omega+3\omega^2+\ldots+n\omega^{n-1}$, then

$\omega S=\qquad \omega+2\omega^2+\ldots+(n-1)\omega^{n-1}+n\omega^n$

From the above two relations we get $S=\frac{n}{\omega-1}$

Now $\sum_{r=1}^{n}(ar+b)\omega^{r-1}=a\sum_{r=1}^{n}r\,\omega^{r-1}+b\sum_{r=1}^{n}\omega^{r-1}$

Or $\sum_{r=1}^{n}(ar+b)\omega^{r-1}=\frac{an}{\omega-1}.$

Q.19 (b)

$(Z+ab)^3=a^3\Rightarrow Z=a-ab,\ a\omega-ab$ & $a\omega^2-ab.$

Now side length $|(a-ab)-(a\omega-ab)|=|a(1-\omega)|$ i.e. $\sqrt{3}|a|.$

Q.20 (d)

$\left|\omega Z-1-\omega^2\right|=a \Rightarrow |Z+1|=a.$

Given $|Z+1|=a$ & $|Z-1|\le 2.$

Now $\left||Z+1|-2\right|\le|Z-1|\Rightarrow|a-2|\le 2$ or $0\le a\le 4.$

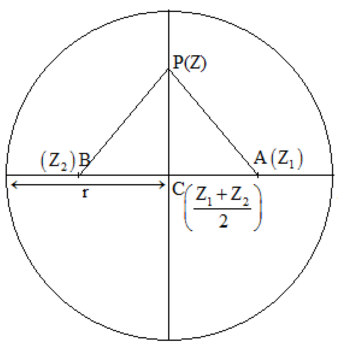
Q.21 (a)

$\left|Z^2+2Z\cos\alpha\right|\le|Z|^2+2|Z||\cos\alpha|$

Now $|Z|<\sqrt{2}-1$ & $\cos\alpha\le 1,$ hence $\left|Z^2+2Z\cos\alpha\right|<\left(\sqrt{2}-1\right)^2+2\left(\sqrt{2}-1\right).$

Or $\left|Z^2+2Z\cos\alpha\right|<1.$

Q.22 (b)



Consider a circle having center at $C\left(\dfrac{Z_1+Z_2}{2}\right)$ and radius r.

Now $A(Z_1)$ & $B(Z_2)$ will be two points on a diameter such that AC = BC.
Also P(Z) will be a point on the perpendicular diameter as given PA = PB.
Clearly area will be maximum when CP = r.

Hence max. area = $\dfrac{1}{2}|Z_1-Z_2|r$.

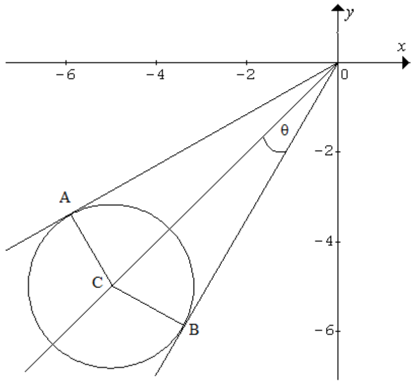
Q.23 (d)

$|Z_2+iZ_1|=|Z_2|+|i\,Z_1|\Rightarrow \arg(Z_2)=\arg(i\,Z_1)$ or $\arg(Z_2)-\arg(Z_1)=\dfrac{\pi}{2}.$

Let $Z_1=3$ & $Z_2=4i,$ then $\dfrac{Z_2-iZ_1}{1-i}=\dfrac{i(1+i)}{2}$ or $\dfrac{-1+i}{2}$

$$\text{Area}=\frac{1}{2}\times\left\|\begin{matrix}1 & 3 & 0\\ 1 & 0 & 4\\ 1 & -\frac{1}{2} & \frac{1}{2}\end{matrix}\right\|=\frac{25}{4}.$$

Q.24 (a)



$|Z+5+5i| \le \frac{5\sqrt{3}-5}{2}$ represents a circle with center at $(-5, -5)$ and radius $\frac{5\sqrt{3}-5}{2}$.

Now OC = $5\sqrt{2}$ & BC = $\frac{5\sqrt{3}-5}{2}$, thus

$\sin\theta = \frac{\sqrt{3}-1}{2\sqrt{2}}$ or $\theta = \frac{\pi}{12}$.

Now angle made by OC with positive real axis is $\frac{5\pi}{4}$, therefore angle made by OB & OC with positive real axis are $\frac{4\pi}{3}$ & $\frac{7\pi}{6}$.

Hence least arg(Z) = $-\frac{5\pi}{6}$.

Q.25 (a)

$$\text{Case I}: |Z-1| < |Z+1| \,\&\, |Z| = |Z-1|$$
$$\text{Case II}: |Z-1| > |Z+1| \,\&\, |Z| = |Z+1|$$

$\Rightarrow x > 0$, then $x = \frac{1}{2}$ & $x < 0$, then $x = -\frac{1}{2}$.

Now $Z + \bar{Z} = 2\,\mathrm{Re}(Z)$, thus $Z + \bar{Z} = 1$ or -1.

Q.26 (b)

$$\arg\left(\frac{Z_1 - \frac{Z}{|Z|}}{\frac{Z}{|Z|}}\right) = \frac{\pi}{2} \Rightarrow \frac{Z_1 - \frac{Z}{|Z|}}{\frac{Z}{|Z|}} = \left|Z_1 - \frac{Z}{|Z|}\right| e^{-\frac{\pi}{2}i}$$

$$\Rightarrow Z_1 - \frac{Z}{|Z|} = 3i\frac{Z}{|Z|} \quad \text{or} \quad Z_1 = (3i+1)\frac{Z}{|Z|}.$$

Hence $|Z_1| = \sqrt{10}$.

Q.27 (b)

The required complex vector will be $\frac{\lambda}{2}\left(\frac{Z_1}{|Z_1|} + \frac{Z_2}{|Z_2|}\right)$ i.e. $\frac{\lambda}{2}\left(\frac{3+\sqrt{3}i}{2\sqrt{3}} + \frac{2\sqrt{3}+6i}{4\sqrt{3}}\right)$.

Hence any complex number of form $\mu(1+i)$ will lie along the angle bisector.

Q.28 (a)

$$|Z-2+2i| \le \big||Z|-|2-2i|\big| \Rightarrow -1 \le |Z|-2\sqrt{2} \le 1.$$

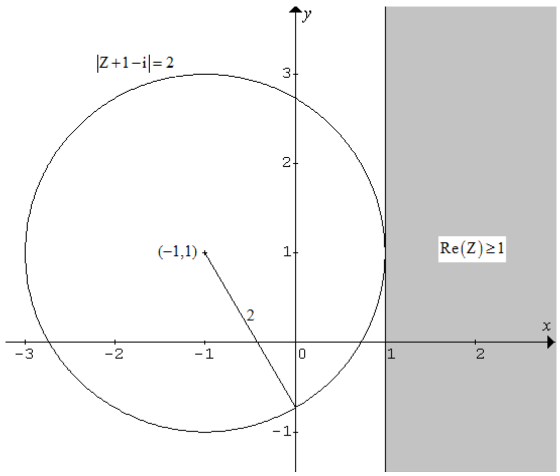
Hence least value of $|Z|$ is $2\sqrt{2}-1$.

Also arg(Z) = arg(2 – 2i).

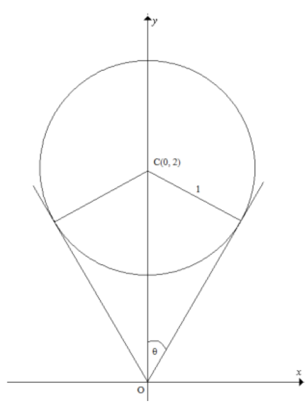
$$\therefore Z = \frac{2\sqrt{2}-1}{\sqrt{2}}(1-i).$$

Q.29 (b)

Refer the adjoining figure.



Q.30 (a)

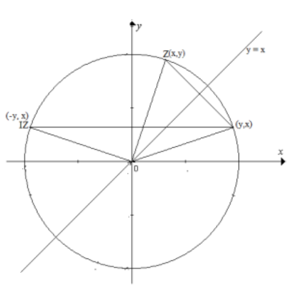


As shown in figure range of arg(Z) will be

from $\frac{\pi}{2}-\theta$ to $\frac{\pi}{2}+\theta$, where $\sin\theta = \frac{1}{2}$ i.e. $\theta = \frac{\pi}{6}$.

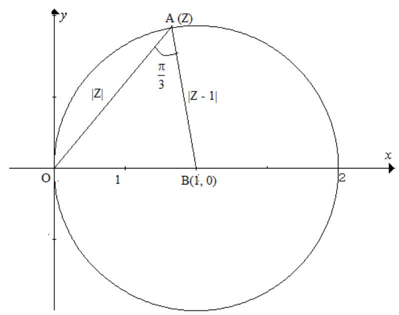
Hence $\text{Arg}(\alpha) \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$.

Q.31 (b)



Rotation of Z(x + iy) about the origin gives iZ(–y + ix).
Then reflection in Imaginary-Axis gives (y + ix),
which is equivalent to reflection of Z in the line x = y.
Hence T_1 is equivalent to composite of T_2 & T_3.

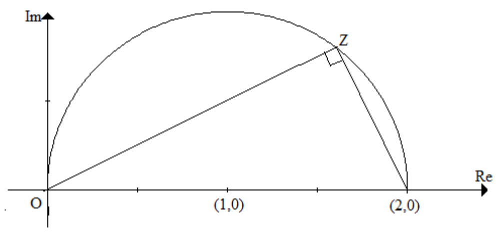
Q.32 (d)



Refer the adjoining figure.
By cosine formula,

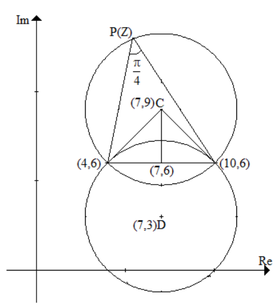
$$\cos\frac{\pi}{3}=\frac{|Z|^2+1-1}{2|Z|}\Rightarrow|Z|^2=|Z|\text{ or }|Z|=1.$$

Q.33 (c)



Refer the adjoining figure.

Q.34 (c)



Angles in same segment of a circle are equal, hence Z will move on major arc of two circles passing through (4, 6) & (10, 6) and of radius $3\sqrt{2}$ as shown in adjoining figure.

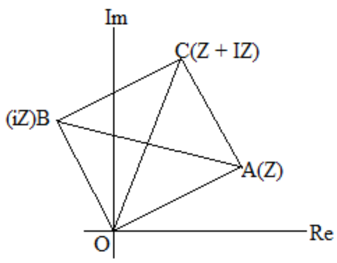
Q.35 (d)

$\cos\frac{2k\pi}{11}+i\sin\frac{2k\pi}{12}$ is 12th root of unity for $k=0,1,2,\ldots,11$.

Now $\sum_{k=0}^{11}\left(\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}\right)=0$ & $\sum_{k=0}^{11}\left(\sin\frac{2k\pi}{n}-i\cos\frac{2k\pi}{n}\right)=-i\sum_{k=0}^{11}\left(\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}\right)$,

hence $\sum_{k=1}^{11}\left(\sin\frac{2k\pi}{n}-i\cos\frac{2k\pi}{n}\right)=i.$

Q.36 (d)



$A(Z) \& B(iZ)$ are such that $OA \perp OB$.

Also $C(Z+iZ)$ will be such that OC is diagonal of Square OACB as shown in adjoining figure.

Hence required area is $\frac{1}{2}|Z|^2$.

Q.37 (c)

Let $P(Z)=(Z-1-i)(Z-1+i)Q(Z)+aZ+b$

Now $P(1+i)=3+4i \Rightarrow (1+i)a+b=3+4i \ ...(i)$

& $P(1ii)=3+4i \Rightarrow (1-i)a+b=4-3i \ ...(ii)$

From (i) & (ii)

$a=\left(\frac{7+i}{2}\right), b=0.$

Q.38 (a)

Note that triangle AOB is right angled isosceles triangle, hence C will be midpoint of AB.

Q.39 (b)

$(a+ib)^n=(a-ib)^n \Rightarrow e^{i(n\theta)}=e^{-i(n\theta)}$, where $\theta=\tan^{-1}\frac{b}{a}$

$\Rightarrow e^{i(2n\theta)}=1 \Rightarrow \tan^{-1}\frac{b}{a}=\frac{\pi}{n} \Rightarrow \frac{b}{a}=\tan\frac{\pi}{n}$

Clearly least positive integral value of n is 3 such that $\frac{b}{a}$ is defined and not zero.

Q.40 (a)

$|2Z_1+Z_2| \le 2|Z_1|+|Z_2| \Rightarrow |2Z_1+Z_2| \le 4.$