APPLICATIONS OF DERIVATIVES
EXERCISE 3

1. Since the curve $y = ax^3 + bx^2 + cx + 5$ touches x-axis at P(-2,0) then x-axis is the tangent at (-2,0). The curve meets y-axis in (0,5). We have $\frac{dy}{dx} = 3ax^2 + 2bx + c$

$\therefore \quad \left.\frac{dy}{dx}\right|_{(0,5)} = 0 + 0 + c = 3$ (given)

$\therefore \quad c = 3$ ...(1)

and $\left.\frac{dy}{dx}\right|_{(-2,0)} = 0 \quad \Rightarrow \quad 12a - 4b + c = 0$

$\Rightarrow \quad 12a - 4b + 3 = 0$ {From (1)} ...(2)

and $(-2,0)$ lies on the curve then

$0 = -8a + 4b - 2c + 5 \quad \Rightarrow \quad 0 = -8a + 4b - 1$ $(\because c = 3)$

$\Rightarrow \quad 8a - 4b + 1 = 0$ ...(3)

From (2) and (3) we get $a = -\frac{1}{2}, b = -\frac{3}{4}$ and $c = 3.$`

2. $x^3 - 3xy^2 + 2 = 0$

$\therefore \quad 3x^2 - 3y^2 - 6xy\frac{dy}{dx} = 0$

$\therefore \quad x^2 - y^2 = 2xy\frac{dy}{dx}$

$\therefore \quad \frac{dy}{dx} = \frac{x^2 - y^2}{2xy} = m_1$

$3x^2y - y^3 + 2 = 0$

$\therefore \quad 6xy + 3x^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx}\left(x^2 - y^2\right) = -2xy$

$\therefore \quad \frac{dy}{dx} = \frac{-2xy}{x^2 - y^2} = m_2$

$m_1 m_2 = -1$

3. $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$

$\therefore \quad \frac{nx^{n-1}}{a^n} + \frac{ny^{n-1}}{b^n}\frac{dy}{dx} = 0$

$$\therefore \quad \frac{dy}{dx} = -\frac{b^n x^{n-1}}{a^n y^{n-1}}$$

At (a, b)

$$\frac{dy}{dx} = -\frac{b}{a}$$

$$\therefore \quad b(x-a) + a(y-b) = 0$$

$$\therefore \quad bx + ay = 2ab$$

$$\therefore \quad \frac{x}{a} + \frac{y}{a} = 2.$$

4. $ay^2 = (x+b)^3$

$$2ay\frac{dy}{dx} = 3(x+b)^2$$

$$\frac{dy}{dx} = \frac{3(x+b)^2}{2ay}$$

$$ST = \frac{2ay^2}{3(x+b)^2}, \quad SN = \frac{3(x+b)^2}{2a}$$

$$\frac{(ST)^2}{SN} = \frac{4a^2y^4}{3(x+b)^4} \times \frac{2a}{3(x+b)^2}$$

$$= \frac{8a(ay^2)^2}{27(ay^2)^2} = \frac{8a}{27}.$$

5. $y = (1+x)^y + \sin^{-1}(\sin^2 x)$

if $\quad x = 0 \;;\; y = 1$

now $\quad y = e^{y\, ln(1+x)} + \sin^{-1}(\sin^2 x)$

differentiating $\quad \frac{dy}{dx} = (1+x)^y\left[\frac{y}{1+x} + \ln(1+x)\frac{dy}{dx}\right] + \frac{\sin 2x}{\sqrt{1-\sin^4 x}}$

put $x = 0$ and $y = 1$

$$\frac{dy}{dx} = 1$$

slope of normal = – 1

$y - 1 = -1(x - 0)$

$x + y - 1 = 0$

6. $2y = -5x^5 + 10x^3 - x - 6$

$$\therefore \quad \frac{dy}{dx} = -\frac{25}{2}x^4 + 15x^2 - \frac{1}{2}$$

At $x = 0, m_N = -\frac{dx}{dy} = 2$

$\therefore$ $y + 3 = 2x$

$\therefore$ $y = 2x - 3$

$\therefore$ $4x - 6 = -5x^5 + 10x^3 - x - 6$

$\therefore$ $5x^5 - 10x^3 + 5x = 0$

$\therefore$ $x\left(x^4 - 2x + 1\right) = 0$

$\therefore$ $x\left(x+1\right)^2\left(x-1\right)^2 = 0$

$\therefore$ $x + 1 = 0$ or $x - 1 = 0$

$\therefore$ $x = \pm 1$

$x = 1 \Rightarrow y = -1$

$x = -1 \Rightarrow y = -5$

$m_{x=1} = 2 = m_{x=-1}$

$\therefore$ Equation of tangents are

$y + 5 = 2\left(x+1\right)$ and $y + 1 = 2\left(x-1\right)$

i.e. $y = 2x - 3$ i.e. the normal becomes the tangent.

7. $x^3 + y^3 = a^3$

$\therefore$ $3x^2 + 3y^2\frac{dy}{dx} = 0$

$\therefore$ $\frac{dy}{dx} = -\frac{x^2}{y^2}$

$\therefore$ $\frac{y - y_1}{x - x_1} = -\frac{x_1^2}{y_1^2}$

$\therefore$ $y_1^2 y - y_1^3 = -x_1^2 x + x_1^3$

$\therefore$ $y_1^2 y + x_1^2 x = a^3$ $\qquad x_1^3 + y_1^3 = a^3$

Now, $y_1^2 y_2 + x_1^2 x_2 = y_1^3 + x_1^3$ [Given]

$\therefore$ $y_1^2\left(y_2 - y_1\right) = x_1^2\left(x_1 - x_2\right)$

$x_2^3 + y_2^3 = a^3$ By solving

We have,

$x_1^3 + y_1^3 = a^3$

$3x_1^2 x_2 + 3y_1^2 y_2 = 3a^3$

$3x_1 x_2^2 + 3y_1 y_2^2 = 3a^3$

$x_2^3 + y_2^3 = a^3$

$\therefore \quad (x_1 + x_2)^3 + (y_1 + y_2)^3 = (2a)^3$

8. Given $\dfrac{x^2}{a^2 + k_1} + \dfrac{y^2}{b^2 + k_1} = 1$ ...(1)

and $\dfrac{x^2}{a^2 + k_2} + \dfrac{y^2}{b^2 + k_2} = 1$ ...(2)

Subtracting (2) from (1), we get

$$x^2\left(\frac{1}{a^2 + k_1} - \frac{1}{a^2 + k_2}\right) + y^2\left(\frac{1}{b^2 + k_1} - \frac{1}{b^2 + k_2}\right) = 0$$

$$\Rightarrow \quad x^2\left(\frac{k_2 - k_1}{(a^2 + k_1)(a^2 + k_2)}\right) + y^2\left(\frac{k_2 - k_1}{(b^2 + k_1)(b^2 + k_2)}\right) = 0$$

$$\therefore \quad \frac{x^2}{y^2} = -\frac{(a^2 + k_1)(a^2 + k_2)}{(b^2 + k_1)(b^2 + k_2)} \quad ...(3)$$

Now form (1), $\dfrac{2x}{(a^2 + k_1)} + \dfrac{2y}{(b^2 + k_1)}\dfrac{dy}{dx} = 0$

$$\therefore \quad \frac{dy}{dx} = -\frac{x(b^2 + k_1)}{y(a^2 + k_1)} = m_1 \text{ (say)}$$

Similarly form (2), $\dfrac{dy}{dx} = -\dfrac{x(b^2 + k_2)}{y(a^2 + k_2)} = m_2$ (say)

$$\therefore \quad m_1 m_2 = \frac{x^2(b^2 + k_1)(b^2 + k_2)}{y^2(a^2 + k_1)(a^2 + k_2)} \quad \{\text{From (3)}\}$$

$$= -1$$

Hence given curves intersect orthogonally.

9. Let $P_1(t_1, t_1^3)$ is a point on the curve $y = x^3$

$$\therefore \quad \left.\frac{dy}{dx}\right|_{(t_1, t_1^3)} = 3t_1^2$$

Tangent at P_1 is $y - t_1^3 = 3t_1^2(x - t_1)$ ...(1)

The intersection of (1) and $y = x^3$

$\Rightarrow \quad x^3 - t_1^3 = 3t_1^2(x - t_1) \quad \Rightarrow \quad (x - t_1)(x^2 + xt_1 + t_1^2) - 3t_1^2(x - t_1) = 0$

$\Rightarrow \quad (x - t_1)^2(x + 2t_1) = 0$

If $P_2(t_2, t_2^3)$, then $(t_2 - t_1)^2(t_2 + 2t_1) = 0$

$\therefore \quad t_2 = -2t_1 \ (t_2 \neq t_1)$

Similarly, the tangent at P_2 will meet the curve at the point $P_3\left(t_3\ t_3^3\right)$ when $t_3=-2t_2=4t_1$ and so on.

The abscissae of $P_1, P_2, \ldots\ldots, P_n$ are $t_1, -2t_1, 4t_1, \ldots, (-2)^{n-1}\, t_1$ in G.P.

$$\therefore \quad \frac{t_2}{t_1}=\frac{t_3}{t_2}=\frac{t_4}{t_3}=\ldots=-2 \text{ (r say)}$$

$$\therefore \quad t_2=t_1 r, t_3=t_2 r \text{ and } t_4=t_3 r$$

$$\therefore \quad \text{Area of } \Delta P_1P_2P_3=\frac{1}{2}\begin{vmatrix} t_1 & t_1^3 & 1 \\ t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \end{vmatrix}$$

$$\text{and} \quad \text{Area of } \Delta P_2P_3P_4=\frac{1}{2}\begin{vmatrix} t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \\ t_4 & t_4^3 & 1 \end{vmatrix}=\frac{1}{2}\begin{vmatrix} r\,t_1 & r^3\,t_1^3 & 1 \\ r\,t_2 & r^3\,t_2^3 & 1 \\ r\,t_3 & r^3\,t_3^3 & 1 \end{vmatrix}=r^4 \text{ (Area of } \Delta P_1P_2P_3\text{)}$$

$$\therefore \quad \frac{\text{Area of}\left(\Delta P_1P_2P_3\right)}{\text{Area of}\left(\Delta P_2P_3P_4\right)}=\frac{1}{r^4}=\frac{1}{(-2)^4}=\frac{1}{16}.$$

10. Given curve is $x^3+y^3=c^3$

$$\therefore \quad \frac{dy}{dx}=-\frac{x^2}{y^2} \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{(a,b)}=-\frac{a^2}{b^2} \qquad \ldots(1)$$

and tangent at (a,b) cuts the curve again at $\left(a_1, b_1\right)$

$$\therefore \quad \text{Slope of tangent}=\frac{b_1-b}{a_1-a}=-\frac{a^2}{b^2} \qquad \{\text{From (1)}\} \qquad \ldots(1)$$

Also $a^3+n^3=c^2$ ...(2)

and $a_1^3+b_1^3=c^3$ ...(3)

subtracting (3) from (2), we get $\left(a^3-a_1^3\right)+\left(b^3-b_1^3\right)=0$

$$\Rightarrow \quad \left(a-a_1\right)\left(a^2+a\,a_1+a_1^2\right)+\left(b-b_1\right)\left(b^2+b\,b_1+b_1^2\right)=0$$

$$\Rightarrow \quad \frac{b_1-b}{a_1-a}=-\frac{a^2+aa_1+a_1^2}{b^2+bb_1+b_1^2} \qquad \ldots(4)$$

From (1) and (4), $$-\frac{a^2}{b^2}=-\frac{a^2+aa_1+a_1^2}{b^2+bb_1+b_1^2}$$

$$\Rightarrow \quad a^2b^2+a^2bb_1+a^2b_1^2=a^2b^2+ab^2a_1+a_1^2b^2$$

$$\Rightarrow \quad ab\left(ab_1-ba_1\right)+a^2b_1^2+a_1^2b^2=0$$

$$\Rightarrow \quad ab\left(ab_1-ba_1\right)+\left(ab_1+a_1b\right)\left(ab_1-a_1b\right)=0$$

$$\Rightarrow \quad \left(ab_1-a_1b\right)+\left(ab+ab_1+a_1b\right)=0$$

If $ab_1 - a_1b = 0$

then $\frac{a}{a_1} = \frac{b}{b_1} = \frac{\sqrt[3]{a^3 + b^3}}{\sqrt[3]{a_1^3 + b_1^3}} = \frac{c}{c} = 1$ (Law of proportion)

$\therefore$ $a = a_1$ and $b = b_1$ which is impossible.

Hence $ab + ab_1 + a_1b = 0$ or $\frac{a_1}{a} + \frac{b_1}{b} = -1$

11. $|\sin x| + |\cos x| \in \left[1, \sqrt{2}\right]$

$\therefore$ $\left[|\sin x| + |\cos x|\right] = 1$

For $x^2 + y^2 = 5, \frac{dy}{dx} = -\frac{x}{y}$

For $y = 1, x = \pm 2$

$\therefore$ $m_1 = 0, m_2 = \mp 2$

$\therefore$ $\tan\alpha = \left|\frac{\pm 2 - 0}{1 + 0}\right| = 2$

$\therefore$ $\alpha = \tan^{-1}(2)$

12. $x^m y^n = k^{m+n}$

$\therefore$ $\frac{dy}{dx} = -\frac{my}{nx}$

$\therefore$ $\frac{Y - y}{X - x} = -\frac{my}{nx}$

$\therefore$ $nxY - nxy = -myX + mxy$

$\therefore$ $myX + nxY = xy(m + n)$

$X = 0 \Rightarrow Y = \frac{y(m+n)}{n}$

$Y = 0 \Rightarrow X = \frac{x(m+n)}{m}$

Now, $x = \frac{\frac{Rx(m+n)}{m} + 0}{k + 1}$, Where R is ratio in which point of contact divides the segments

$\therefore$ $mR + m = Rm + Rn$

$\therefore$ $R = \frac{m}{n}$

$\therefore$ Ratio $= m : n$

13. $\frac{x^2}{a}+\frac{y^2}{b}=1 \ : \frac{x^2}{a_1}+\frac{y^2}{b_1}=1$

$m_1 = -\frac{bx}{ay}$, $m_2 = -\frac{b_1 x}{a_1 y}$

solving, $x^2 = a, y^2 = b_1$

$\therefore \quad m_m m_2 = \frac{bb_1 a}{aa_1 b_1} = \frac{b}{a_1} = -1$

$\therefore$ Condition for orthogonality is $a_1 + b = 0$.

14. T.P. $\frac{\sin\theta}{\theta} < \frac{\sin(\sin\theta)}{\sin\theta}$

Let $f(x) = \frac{\sin x}{x}$

$\therefore \quad f'(x) = \frac{x\cos x - \sin x}{x^2}$

Let $g(x) = x\cos x - \sin x$

$\therefore \quad g'(x) = -x\sin x < 0$ for $x \in \left(0, \frac{\pi}{2}\right)$

$g(0) = 0$

$\therefore \quad g(x) < 0$ for $x \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad f'(x) < 0$ for $x \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad f(x)$ is M.D. in $\left(0, \frac{\pi}{2}\right)$

Now $\theta > \sin\theta$ for $x \in \left(0, \frac{\pi}{2}\right)$ (as shown in Q.9)

$\therefore \quad f(\theta) < f(\sin\theta)$ Hence provrd.

15. $f(x) = \frac{1 - x + x^2}{1 + x + x^2}$

$\therefore \quad f'(x) = \frac{(2x-1)(x^2+x+1) - (2x+1)(x^2-x+1)}{(x^2+x+1)^2}$

$= \frac{(2x^3+x^2+x-1) - (2x^3-x^2+x+1)}{(x^2+x+1)^2}$

$$= \frac{2(x+1)(x-1)}{(x^2+x+1)^2}$$

Now, $x^2 + x + 1 > 0 \quad \forall\, x \in R$

$\therefore$ $f(x)$ is MI in $(-\infty,-1)\cup(1,\infty)$ and MD in $(-1,1)$.

16. $f(x) = (m+2)x^3 - 3mx^2 + 9mx - 1$

$\therefore$ $f'(x) = 3(m+2)x^2 - 6mx + 9m < 0 \quad \forall\, x \in R$

$\therefore$ $(m+2)x^2 - 2mx + 3m < 0 \quad \forall\, x \in R$

$\therefore$ $m + 2 < 0 \quad \Rightarrow \quad m < -2$

and $4m^2 - 4(m+2)(3m) < 0$

$\therefore$ $m^2 - (m+2)(3m) < 0$

$\therefore$ $-2m^2 - 6m < 0$

$\therefore$ $m(m+3) > 0$

$\therefore$ $m < -3$ as $m < -2$

$\therefore$ $m \in (-\infty, -3)$

17. $\frac{\sin x}{x}$ is MD is proved in Q.10

Now $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$ & $\frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \frac{2}{\pi}$

$\therefore$ $\frac{2}{\pi} < \frac{\sin x}{x} < 1$ for $x \in \left(0, \frac{\pi}{2}\right)$

$\therefore$ $\frac{2x}{\pi} < \sin x < x$ for $x \in \left(0, \frac{\pi}{2}\right)$

18. Refer assignment Q.10

19. $h(x) = f(x) - [f(x)]^2 + [f(x)]^3$

$\therefore$ $h'(x) = f'(x) - 2f(x)[f'(x)] + 3[f(x)]^2 f'(x) > 0$

$\therefore$ $f'(x)\underbrace{\left(3f^2(x) - 2f(x) + 1\right)}_{>0 \text{ for all } f(x)\in R} > 0$

$\therefore$ $f'(x) > 0$ for $h'(x) > 0$

20. $f(x) = \begin{cases} xe^{ax} & , \quad x \le 0 \\ x + ax^2 - x^3 & , \quad x > 0 \end{cases}$, $a > 0$

$f'(x) = e^{ax}(1 + ax), x \le 0$

$1 + 2ax - 3x^2, x > 0$

$f'(x) > 0$

For $x \le 0$

$f'(x) > 0$ for $x \in \left(\frac{1}{a}, 0\right]$

For $x > 0$,

$3x^2 - 2ax - 1 < 0$ for $f'(x) > 0$

$\therefore \quad x \in \left[0, \frac{a + \sqrt{a^2 + 3}}{3}\right)$

$\therefore \quad f(x)$ is monotonically increasing in $x \in \left(\frac{1}{a}, \frac{a + \sqrt{a^2 + 3}}{3}\right)$

21. $f''(x) > 0 \quad \forall\, x \in R$

$Q(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)$

$\therefore \quad Q(x) = 2x\, f'\left(\frac{x^2}{2}\right) - 2xf'(6 - x^2) > 0$

$\therefore \quad 2x\left(f'\left(\frac{x^2}{2}\right) - f'(6 - x^2)\right) > 0$

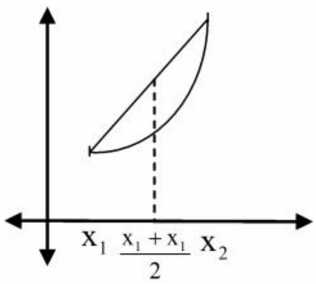
$\therefore \quad x\left(f'\left(\frac{x^2}{2}\right) - f'(6 - x^2)\right) > 0$

$\frac{x^2}{2} = 6 - x^2 \quad \Rightarrow \quad x = \pm 2$

$\therefore \quad Q(x)$ is MI in $(-2, 0) \cup (2, \infty)$ & MD in $(-\infty, -2) \cup (0, 2)$

22. $f'(x) > 0$, $f''(x) > 0$, $x_1 < x_2$

$\therefore$ Graph of f(x) is



By graph

$f\left(\frac{x_1 + x_2}{2}\right) < \frac{f(x_1) + f(x_2)}{2}$

23. $Q(x) = 2f\left(\frac{x}{2}\right) + f(2-x)$

$\therefore \quad Q'(x) = f'\left(\frac{x}{2}\right) - f'(2-x)$

$f''(x) < 0$

$\therefore \quad g'(x) > 0$

$\Rightarrow \quad f'\left(\frac{x}{2}\right) > f'(2-x)$

$\Rightarrow \quad \frac{x}{2} < 2 - x$

$\therefore \quad x < \frac{4}{3}$

$\therefore \quad Q(x)$ increases in $\left(-\infty, \frac{4}{3}\right)$ & decreases in $\left(\frac{4}{3}, \infty\right)$

24. $f(x) = \sin^3 x + \lambda \sin^2 x$

$\therefore \quad f'(x) = 3\sin^2 x \cos x + 2\lambda \sin x \cos x = 0$

$\therefore \quad \cos x\left(3\sin^2 x + 2\lambda \sin x\right) = 0$

$\therefore \quad \cos x \neq 0$ as $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \quad \sin x(3\sin x + 2\lambda) = 0$

$\therefore \quad 3\sin x + 2\lambda = 0$ has a root

$\therefore \quad \sin x = -\frac{2\lambda}{3}$ has a root

$\therefore \quad \lambda \in \left(-\frac{3}{2}, \frac{3}{2}\right) - \{0\}$ as $\sin x = 0$ is already a root.

25. $f(x)$ is 6^{th} degree polynomial

$$\lim_{x \to 0}\left\{1 + \frac{f(x)}{x^3}\right\}^{\frac{1}{x}} = r = e^2$$

$\therefore \quad e^{\lim_{x \to 0} \frac{f(x)}{x^4}} = e^2 \qquad f(x) = ax^6 + bx^5 + cx^4$ & $c = 2$

$\therefore \quad$ Roots are 1 & 2

$2 = \frac{8}{6a} \qquad \therefore \qquad a = \frac{2}{3}$

$\therefore \quad f'(x) = 6ax^5 + 5bx^4 + 8x^3 = 0$

$\therefore \quad 6ax^2 + 5bx + 8 = 0$ has roots 1 & 2

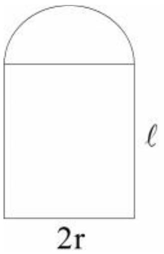
$\therefore \quad -\frac{5b}{6a} = 3$ and $\frac{8}{6a} = 2$

$\therefore \quad a = \frac{2}{3} \qquad b = -3 \times 6 \times \frac{2}{3} \times \frac{1}{5}$

$b = -\frac{12}{5}$

$\therefore \quad f(x) = \frac{2}{3}x^6 - \frac{12}{5}x^5 + 2x^4$

26.



$P = \pi r + 2r + 2\ell + 2r$

$= 4r + 2\ell + \pi r$

$\therefore \quad \frac{dP}{dr} = 4 + \pi + \frac{2d\ell}{dr} = 0$

$\frac{d\ell}{dr} = -\frac{(4+\pi)}{2}$

$L = 3 \times 2r\ell + \frac{\pi r^2}{2}$

$\therefore \quad \frac{dL}{dr} = 0$

$\therefore \quad 6\ell + 6r\left(\frac{4+\pi}{2}\right) + \pi r = 0$

$\therefore \quad 3\ell = 12r + 2\pi r$

$\frac{\ell}{r} = \frac{12 + 2\pi}{3}$

27.

(a) $\frac{dx}{dt} = 6t^2 - 6t + 6$

$\frac{dy}{dt} = 6t^2 + 6t + 6$

$\therefore \quad \frac{dy}{dx} = \frac{6t^2 + 6t + 6}{6t^2 - 6t + 6}$

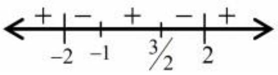
$= \frac{t^2 + t + 1}{t^2 - t + 1}$

Solving it is maximum at $t = 1$ and has max value = 3.

$\therefore$ Point on curve $= (5, 11)$

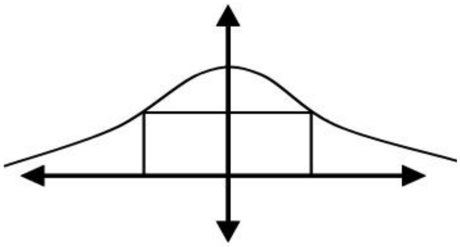
(b) $\frac{dy}{dx} = \frac{12t^2 - 6t - 18}{5t^4 - 15t^2 - 20} = \frac{6}{5}\frac{\left(2t^2 - t - 3\right)}{\left(t^4 - 3t^2 - 4\right)}$

$= \frac{6(2t-3)(t+1)}{5(t+2)(t-2)}$



$\therefore$ $f(x)$ has minimum value at $t = -1$ and max at $t = \frac{3}{2}$

28.



By symmetry, points on x-axis will be $(-a,0)$ and $(a,0)$

$A = 2ae^{-a^2}$

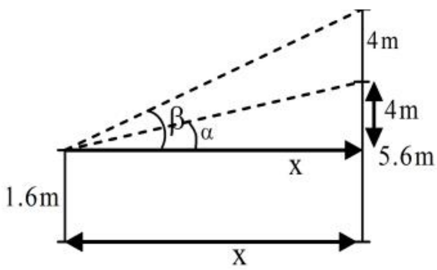
$\therefore \quad \frac{dA}{da} = 2e^{-a^2} - (2a)^2 e^{-a^2} = 0$

$\therefore \quad 2a^2 = 1$

$\therefore \quad a = \pm\frac{1}{\sqrt{2}}$

$\therefore \quad A = \sqrt{2}\, e^{-\frac{1}{2}} = \sqrt{\frac{2}{e}}$

29.



$\tan\alpha = \frac{4}{x}, \tan\beta = \frac{8}{x}$

$\therefore \quad \beta - \alpha = \tan^{-1}\left(\frac{8}{x}\right) - \tan^{-1}\left(\frac{4}{x}\right) = \tan^{-1}\left(\frac{\frac{4}{x}}{1 + \frac{32}{x^2}}\right) = \tan^{-1}\left(\frac{4x}{x^2 + 32}\right)$

$\tan^{-1} x$ is an increasing function

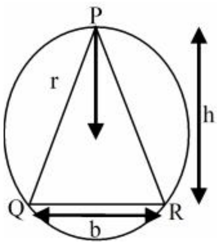
$\therefore \quad \max \text{ of } \dfrac{4x}{x^2+32}$

$\Rightarrow \quad \max \text{ of } \tan^{-1}\left(\dfrac{4x}{x^2+32}\right)$

$f(x)=\dfrac{4x}{x^2+32}$

$\therefore \quad f'(x)=\dfrac{4\left(x^2+32\right)-8x^2}{\left(x^2+32\right)^2}=0$

$\therefore \quad 128=4x^2 \qquad \therefore \quad x=\sqrt{32}\text{ m}=4\sqrt{2}$

31. 

$(h-r)^2+\dfrac{b^2}{4}=r^2$

Solving as in (level 1) Q.12 We get the answer.

32. Q is (1,0)

Let equation of circle wilth centre be

$(x-1)^2+y^2=r^2$

$x^2+y^2=1$

$\therefore \quad -2x+1=r^2-1$

$\therefore \quad x=\dfrac{2-r^2}{2}$

$\therefore \quad y^2=1-\left(\dfrac{2-r^2}{2}\right)^2=1-1+r^2-\dfrac{r^4}{4}$

$\therefore \quad y=r\sqrt{-\dfrac{r^2}{4}+1}$

$\therefore \quad \Delta=\dfrac{1}{4}r^2\sqrt{4-r^2}$

$\therefore \quad \dfrac{d\Delta}{dr}=\dfrac{r}{2}\sqrt{4-r^2}-\dfrac{r^2}{4}\times\dfrac{r}{\sqrt{4-r^2}}=0$

$\therefore \quad \dfrac{r}{2}\sqrt{4-r^2}=\dfrac{r^3}{4\sqrt{4-r^2}}$

$\therefore \quad 2r\left(4-r^2\right)=r^3$

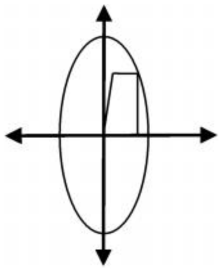
$\therefore \quad r = 0 \quad \text{or} \quad 8 - 2r^2 = r^2$

$\therefore \quad r = \sqrt{\frac{8}{3}}$

$\therefore \quad \sqrt{4 - r^2} = \frac{2}{\sqrt{3}}$

$\therefore \quad \Delta = \frac{1}{4} \times \frac{8}{3} \times \frac{2}{\sqrt{3}} = \frac{4}{3\sqrt{3}}$

33.



$\Delta = \frac{1}{2} \times \beta \times (\alpha + \alpha - \beta)$

$= \frac{1}{2}\beta(2\alpha - \beta)$

$= \alpha\beta - \frac{\beta^2}{2}$

$4\alpha^2 + 3\beta^2 = 12$

$\therefore \quad 8\alpha\frac{d\alpha}{d\beta} + 6\beta = 0$

$\therefore \quad \frac{d\alpha}{d\beta} = -\frac{3\beta}{4\alpha}$

$\therefore \quad \frac{d\Delta}{d\beta} = \alpha - \beta\left(\frac{3\beta}{4\alpha}\right) - \beta = 0 \qquad \therefore \quad 4\alpha^2 - 4\alpha\beta - 3\beta^2 = 0$

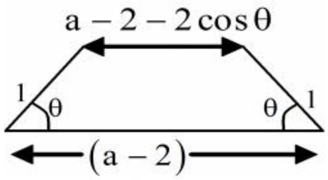
$\therefore \quad (2\alpha + \beta)(2\alpha - 3\beta) = 0 \qquad \text{But } \alpha\beta > 0$

$\therefore \quad 2\alpha = 3\beta \qquad \therefore \quad 4\alpha^2 = 9\beta^2$

$4\alpha^2 + 3\beta^2 = 12$

$\therefore \quad \beta = 1 \text{ and } \alpha = \frac{3}{2} \qquad ....(\alpha, \beta > 0)$

Point is $\left(\frac{3}{2}, 1\right)$

34. 

$h = \sin\theta$

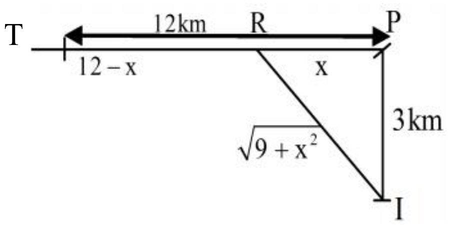
$\therefore \quad A = \frac{1}{2}\sin\theta(2a - 4 - 2\cos\theta)$

$= a\sin\theta - 2\sin\theta - \sin\theta\cos\theta$

$\therefore \quad \frac{dA}{d\theta} = a\cos\theta - 2\cos\theta - \cos^2\theta + \sin^2\theta = 0$

$\therefore \quad 2\cos^2\theta + (2-a)\cos\theta - 1 = 0 \qquad \theta \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad \cos\theta = \frac{a - 2 + \sqrt{(a-2)^2 + 8}}{4}$

35. 

$\therefore$ Rate of cable $= 4000\sqrt{94x^2} + 2000(12 - x)$

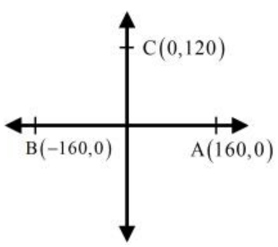
$\therefore \quad \frac{dR}{dx} = \frac{4000x}{\sqrt{9+x^2}} - 2000 = 0$

$\therefore \quad 2x = \sqrt{9 + x^2}$

$\therefore \quad 4x^2 = x^2 + 9$

$\therefore \quad x = \sqrt{3}$ km from P towards T.

36. Solving we get $A \equiv (160, 0), B \equiv (-160, 0), C \equiv (0, 120)$



As ΔABC is an isosceles Δ, by symmetry, godown should be located on u-axis.

Let godown be lovateda at (0,y) , $0 < y < 120$

$\therefore$ Sum of distance from godown is

$s = 120 - y + 2\sqrt{y^2 + 25600}$

$\therefore \quad \frac{ds}{dy} = -1 + \frac{2y}{\sqrt{y^2 + 25600}}$

$\therefore \quad 4y^2 = y^2 + 25600$

$\therefore \quad y^2 = \frac{25600}{3}$

$\therefore \quad y = \frac{160}{\sqrt{3}}$

$\therefore$ Godown should be located at $\left(0, \frac{160}{\sqrt{3}}\right)$.

37. $P = (1200 - x)(725 + x)$

$\therefore \quad \frac{dP}{dx} = -725 - x + 1200 - x = 0$

$\therefore \quad x = \frac{475}{2} = 237.5$

But as number of subscribers is an integer $x = 237$ or 238 (Values are same)

$\therefore$ Subscribers $= 962$ or 963.

38. $g(t) = 3t^4 - 8t^3 - 6t^2 + 24t$

$g'(t) = 12t^3 - 24t^2 - 12t + 24 = 0$

$\therefore \quad t^3 - 2t^2 - t + 2 = 0$

$\therefore \quad (t^2 - 1)(t - 2) = 0$

By examining it has maxima at $t = 1$ and minima at $t = 2$

$\therefore \quad f(x) = 3x^4 - 8x^3 - 6x^2 + 24x \; ; \; 1 \le x < 2$

$h(t) = 3t + \frac{1}{4}\sin^2 \pi t + 2$

$\therefore \quad h'(t) = 3 + \frac{\pi}{4} \times \sin 2\pi t > 0 \qquad \forall\, t$

$\therefore \quad f(x)$ has minima at $x = 2$

$\therefore \quad f(x) = 3x^4 - 8x^3 - 6x^2 + 24x \; ; \; 1 \le x < 2$

$\quad = 3x + \frac{1}{4}\sin^2 \pi x + 2 \; ; \; 2 \le x \le 4$

$f(1) = 3 - 8 - 6 + 24 = 13$

$\lim_{x \to 2^-} f(x) = 48 - 64 - 24 + 48 = 8$

$f(2) = 6 + 2 = 8$

$f(4) = 12 + 2 = 14$

$\therefore$ Greatest value of $f(x)=14$ and least value of $f(x)=8$.

40. Refer Ex. level 1 (Q.21) for method

We get $\frac{b^3-b^2+b-1}{b^2+3b+2}\ge 0$

$\therefore \quad \frac{(b^2+1)(b-1)}{(b+1)(b+2)}\ge 0$

$\therefore \quad b\in(-2,-1)\cup[1,\infty)$.

41. $f(x)=x+\frac{a}{x^2}-2>0 \qquad \forall x\in R$

$\therefore \quad f'(x)=1-\frac{2a}{x^3}=0$

$\Rightarrow \quad x=(2a)^{\frac{1}{3}}$

$\therefore \quad (2a)^{\frac{1}{3}}+\frac{a^{\frac{1}{3}}}{2^{\frac{1}{3}}}-2>0$

$\therefore \quad \left(4^{\frac{1}{3}}+1\right)a^{\frac{1}{3}}>2^{\frac{4}{3}}$

$\therefore \quad a>\frac{2^4}{\left(4^{\frac{1}{3}}+1\right)^3}$

$\left(4^{\frac{1}{3}}+1\right)^3=5+3\left(4^{\frac{2}{3}}+4^{\frac{1}{3}}\right)$

$\therefore \quad a>1$ ....

$\therefore$ Least natural number is 2.

42. $f'(c)=2c-2=-1$

$\therefore \quad c=\frac{1}{2}$

$\therefore \quad x=\frac{1}{2},\ y=\frac{1}{4}-1+3=\frac{9}{4}$

$\therefore \quad y-\frac{9}{4}=\frac{1}{2}-x$

$\therefore \quad x+y=\frac{11}{4}$

i.e. $\quad 4x+4y=11$

43. $\frac{f'(c)}{g'(c)} = \frac{f(1)-f(0)}{g(1)-g(0)} = \frac{4}{2} = 2$

44. $f(x) = x + e^x$

$\therefore \quad f'(x) = 1 + e^x > 0 \quad \forall\, x \in R$

Now, $f(-\infty) = -\infty,\ f(\infty) = \infty$

$\therefore \quad f(x)$ has only one real root.

45. $f(0) = 4\ ,\ f(5) = -1$

$g(x) = \frac{f(x)}{x-1}$

$g(0) = -4\ ,\ g(5) = -\frac{1}{4}$

$\therefore \quad g'(c) = \frac{4 - \frac{1}{4}}{5} = \frac{3}{4}$

46. $f(x)$ and $f'(x)$ should have a common root

$f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots + \frac{x^n}{n!}$

$\therefore \quad f'(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\cdots + \frac{x^{n-1}}{(n-1)!}$

Which do not have a common root.

47. $f(x) = x + \cos x - a$

$f'(x) = 1 - \sin x \geq 0 \quad \forall\, x \in R$

$\therefore \quad f(x)$ is an increasing function

If $a < 1$, $f(0) > 0$

$\therefore$ No positive root

If $a > 1$, $f(0) < 0$

$\therefore$ one positive root.

48. Consider the function $f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + d$

We have $f(0) = d$ and

$f(1) = \frac{a}{3} + \frac{b}{2} + c + d = \frac{2a + 3b + 6c}{6} + d = 0 + d = d \qquad (\because\ 2a + 3b + 6c = 0)$

Therefore, 0 and 1 are the roots of the polynomial f(x).

Consequently, there exists at least one root of the polynomial $f'(x) = ax^2 + bx + c$ lying between 0 and 1.