APPLICATIONS OF DERIVATIVES
EXERCISE 2(C)

1. $y = ax^3 + bx^2 + cx + 5$

It has repeated root (-2)

$\frac{dy}{dx} = 3ax^2 + 2bx + c$

$y(-2) = 0$

$\therefore \quad -8a + 4b - 2c + 5 = 0$

$y'(-2) = 0$

$\therefore \quad 12a - 4b + c = 0$

$y'(0) = 3$

$\therefore \quad c = 3$

$\therefore \quad 12a - 4b + 3 = 0$ and $-8a + 4b - 1 = 0$

$\therefore \quad a = -\frac{1}{2}$ and $b = -\frac{3}{4}, c = 3$

2. $xy = 4, x^2 + y^2 = 8$

$m_1 = -\frac{y}{x}, m_2 = -\frac{x}{y}$

Clearly curves intersect at $\left(\pm 2\sqrt{2}, \pm 2\sqrt{2}\right)$

Here , $m_1 = m_2$

$\therefore$ Curves touch each other.

3. $y = \cos(x + y)$

$\therefore \quad \frac{dy}{dx} = -\sin(x + y)\left[1 + \frac{dy}{dx}\right]$

$\therefore \quad \frac{dy}{dx} = \frac{-\sin(x+y)}{1 + \sin(x+y)} = -\frac{1}{2}$ .....[Given]

$\therefore \quad 2\sin(x + y) = 1 + 1\sin(x + y)$

$\therefore \quad \sin(x + y) = 1$

$\therefore \quad x + y = (4n + 1)\frac{\pi}{2}$ and $y = \cos\left(\frac{(4n+1)\pi}{2}\right) = 0$

$\therefore \quad x = \frac{(4n+1)\pi}{2}$

But $-2\pi \le x \le 2\pi$

$\therefore \quad x = \frac{-3\pi}{2} \quad \text{or} \quad \frac{\pi}{2}$

$\therefore$ Equation of tangents are $x + 2y = \frac{\pi}{2}$ and $x + 2y = \frac{-3\pi}{2}$

4. $x^2 = 4y$

$\frac{dy}{dx} = \frac{x}{2}$

$\therefore \quad -\frac{dx}{dy} = -\frac{2}{x} = m_N$

$\therefore \quad m_N = -2$

$y - 2 = -2(x - 1)$

$\therefore \quad 2x + y = 4$

5. Given curve is $y = \sin x$ ...(1)

$\therefore \quad \frac{dy}{dx} = \cos x$

Equation ot tangent at (x, y) is $Y - y = \frac{dy}{dx}(X - x)$

or $Y - y = \cos x(X - x)$ ...(2)

Since tangent are drawn fro origin then origin (0,0) lies on (2)

$\therefore \quad 0 - y = \cos(0 - x) \quad \text{or} \quad \cos x = \frac{y}{x}$ ...(2)

From (1) and (2),

$\sin^2 x + \cos^2 x = y^2 + \frac{y^2}{x^2}$

$\Rightarrow \quad 1 = \frac{x^2y^2 + y^2}{x^2} \quad \Rightarrow \quad x^2y^2 = x^2 - y^2.$

6. $\tan\theta = n\,a^{1-n}\,x^{n-1}$

$\therefore \quad SN = y\tan\theta$

$= na^{2-2n}\,x^{2n-1}$

$= \text{Constant}$

$\therefore \quad n = \frac{1}{2}$

7. Given $x^{m+n} = a^{m-n}.y^{2n}$ ...(1)

Taking logarithm of both sides, we get $(m + n)\ln x = (m - n)\ln a + 2n\ln y$

Differentiating of both sides w.r.t. x, we get $\frac{(m+n)}{x} = 0 + \frac{2n}{y}\frac{dy}{dx}$

$\therefore \quad \frac{dy}{dx} = \frac{(m+n)}{2n} \cdot \frac{y}{x}$

Now $\frac{(\text{Sub-tangent})^m}{(\text{Sub-normal})^n} = \frac{\left(y\frac{dx}{dy}\right)^m}{\left(y\frac{dy}{dx}\right)^n} = \frac{y^{m-n}}{\left(\frac{dy}{dx}\right)^{m+n}}$

$$= \frac{y^{m-n}}{\left\{\frac{(m+n)}{2n} \cdot \frac{y}{x}\right\}^{m+n}} = \frac{x^{m+n}}{\left(\frac{m+n}{2n}\right)^{m+n} .y^{2n}} = \frac{a^{m-n}}{\left(\frac{m+n}{2n}\right)^{m+n}} \qquad \{\text{from } (1)\}$$

$(\text{Sub-tangent})^m \propto (\text{Sub-normal})^n$

8. $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 5$

$f'(x) = 4x^3 - 24x^2 + 44x - 24 > 0$

$\therefore \quad x^3 - 6x^2 + 11x - 6 > 0$

$\therefore \quad (x-1)(x-2)(x-3) > 0 \qquad \therefore \quad x \in (1,2) \cup (3,\infty)$

9. $f(x) = x - 2\sin x$

$f'(x) = 1 - 2\cos x > 0$

$\therefore \quad \cos x < \frac{1}{2}$ in $0 \le x \le 2\pi$

$\therefore \quad x \in \left(\frac{\pi}{3}, \frac{5\pi}{3}\right).$

10. $f(x) = 2x^2 - \ell n|x|$

$\therefore \quad f'(x) = 4x - \frac{1}{x}$

$= \frac{(2x+1)(2x-1)}{x}$

M.I. in $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$ & M.D. in $\left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

11. $f(x) = \sin x + \cos x \qquad x \in [0, 2\pi]$

$\therefore \quad f'(x) = \cos x - \sin x$

$\therefore \quad f(x)$ is M I in $\left[\left(0,\frac{\pi}{4}\right)\cup\left(\frac{5\pi}{4},2\pi\right)\right]$ & MD in $\left(\frac{\pi}{4},\frac{5\pi}{4}\right)$.

12. $g(x)=f(x)+f(1-x) \qquad x\in[0,1]$

$g'(x)=f'(x)-f'(1-x)$

$f''(x)<0$

$\therefore \quad f'(x)>f'(1-x)$

$\Rightarrow \quad x<1-x$

$\Rightarrow \quad$ in $x\in\left[0,\frac{1}{2}\right) \quad g'(x)>0 \quad \& \quad g'(x)<0$ in $x\in\left(\frac{1}{2},1\right]$

13. TPT $\quad \ell n(1+x)<0 \quad ; x>0$

i.e. $\quad x>\ell n(1+x)$

Let $\quad f(x)=x-\ell n(1+x)$

$\therefore \quad f'(x)=1-\frac{1}{1+x}$

$$=\frac{x}{1+x}>0 \qquad \forall\ x>0.$$

Now, $\quad f(0)=0$

$\therefore \quad f(x)>0 \qquad \forall\, x>0$

$\therefore \quad x>\ell n(1+x)$

14. $ax^2+\frac{b}{x}\geq c$

Let $\quad f(x)=ax^2+\frac{b}{x}$

$\therefore \quad f'(x)=2ax-\frac{b}{x^2}$

$$=\frac{2ax^3-b}{x^2}=0 \text{ at point of minima.}$$

as $2ax^3-b$ is an increasing function $\forall\ a>0$.

$\therefore \quad x=\left(\frac{b}{2a}\right)^{\frac{1}{3}}$

$\therefore \quad ax^2+\frac{b}{x}=a\left(\frac{b^2}{4a^2}\right)^{\frac{1}{3}}+b\times\left(\frac{2a}{b}\right)^{\frac{1}{3}}$

$$= \frac{a^{\frac{1}{3}} b^{\frac{2}{3}}}{4^{\frac{1}{3}}} + 2^{\frac{1}{3}} b^{\frac{2}{3}} a^{\frac{1}{3}}$$

$$= \frac{3 a^{\frac{1}{3}} b^{\frac{2}{3}}}{4^{\frac{1}{3}}} \geq c$$

$\therefore$ $27ab^2 \geq 4c^3$ [Cubing both sides]
as a,b,c are positive.

15. $0 < x_1 < x_2 < \frac{®}{2}$

T.P. $\frac{\tan x_2}{\tan x_1} > \frac{x_1}{x_2}$

i.e. T.P. $x_2 \tan x_2 > x_1 \tan x_1$

Let $f(x) = x \tan x$

$\therefore$ $f'(x) = \tan x + x \sec^2 x > 0 \quad \forall x \in \left(0, \frac{\pi}{2}\right)$

as $\tan x$ & $x \sec^2 x$ are positive

16. P.T. $x > \sin x > x - \frac{x^3}{6}$ for $0 < x \leq \frac{\pi}{2}$

Let $f(x) = x - \sin x$

$\therefore$ $f'(x) = 1 - \cos x > 0$ for $0 < x \leq \frac{\pi}{2}$

$\therefore$ $x > \sin x$

Let $g(x) = \sin x - x + \frac{x^3}{6}$

$\therefore$ $g'(x) = \cos x - 1 + \frac{x^2}{2}$

$\therefore$ $g''(x) = x - \sin x > 0$ for $0 < x \leq \frac{\pi}{2}$ as shown

$\therefore$ $g'(x) = 0$ for $x = 0$ & increases as x increases

$\therefore$ $g'(x) > 0$ for $0 < x \leq \frac{\pi}{2}$

$g(x) = 0$ for $x = 0$

$\therefore$ $g(x) > 0$ for $0 < x \leq \frac{\pi}{2}$

$\therefore \quad \sin x > x - \frac{x^3}{6}$

17. $f(x) = \frac{1}{8}\ell n\, x - bx + x^2$

$\therefore \quad f'(x) = 2x - b + \frac{1}{8x} = 0$

$\therefore \quad 2x^2 - bx + 1 = 0$

$\therefore \quad x = \frac{b \pm \sqrt{b^2 - 4}}{4} \quad$ If $b > 2$

$f''(x) = 2 - \frac{1}{8x} \qquad > 0$ if $x = \frac{b + \sqrt{b^2 - 4}}{4}$ and

< 0 if $x = \frac{b - \sqrt{b^2 - 4}}{4}$

$\therefore$ It has maxima at $x = \frac{b - \sqrt{b^2 - 4}}{4}$ and

minima at $x = \frac{b + \sqrt{b^2 - 4}}{4}$.

18. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Equation of tangent is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

$x = 0 \quad \Rightarrow \quad y = b\,\mathrm{cosec}\,\theta$ and

$y = 0 \quad \Rightarrow \quad x = a\sec\theta$

Now, length of intercept $= \sqrt{(a\sec\theta - 0)^2 + (b\,\mathrm{cosec}\,\theta - 0)^2} = \sqrt{a^2\sec^2\theta + b^2\,\mathrm{cosec}^2\theta}$

Which has minimum value $a + b$ is

$a^2\sec^2 x + b^2\,\mathrm{cosec}^2 x = f(x)$

$f'(x) = 2a^2\sec^2 x\tan x - 2b^2\,\mathrm{cosec}^2 x\cot x = 0$

$f(x)$ can have minima only as maxima is ∞.

$\therefore \quad a^2\sec^2 x\tan x = b^2\,\mathrm{cosec}^2 x\cot x$

$\therefore \quad a^2\frac{\sin x}{\cos^3 x} = b^2\frac{\cos x}{\sin^3 x}$

$\therefore \quad \tan^4 x = \frac{b^2}{a^2}$

$\therefore \quad \tan^2 x = \left|\frac{b}{a}\right|, \ \cot^2 x = \left|\frac{a}{b}\right|$

For $a, b > 0$

$$\sec^2 x = \frac{a+b}{a}, \operatorname{cosec}^2 x = \frac{a+b}{b}$$

$\therefore$ $a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x$

$$= a^2 + ab + ab + b^2 = (a+b)^2$$

19. $S = \pi r\sqrt{r^2 + b^2} + \pi r^2$

$$= \pi r^2 \operatorname{cosec}\alpha + \pi r^2 = \text{constant}$$

$$V = \frac{1}{3}\pi r^3 \cot\alpha \qquad \therefore \qquad r = \sqrt{\frac{S}{\pi(\operatorname{cosec}\alpha + 1)}}$$

$$= \frac{1}{3}\frac{\pi S^{\frac{3}{2}} \cot\alpha}{\pi^{\frac{3}{2}}(\operatorname{cosec}\alpha + 1)^{\frac{3}{2}}}$$

$$V = \sqrt{\frac{S^3}{9\pi}} \times \frac{(\cos\alpha)\sqrt{\sin\alpha}}{(1+\sin\alpha)^{\frac{3}{2}}}$$

$$= \frac{(\cos\alpha)\sqrt{\sin\alpha}(1-\sin\alpha)^{\frac{3}{2}}}{\cos^3\alpha} \times \sqrt{\frac{S^3}{9\pi}}$$

$$= \sqrt{\sin\alpha}(1-\sin\alpha)^{\frac{3}{2}} \sec^2\alpha \times \sqrt{\frac{S^3}{9\pi}}$$

$$\therefore \quad \frac{dV}{d\alpha} = \sqrt{\frac{S^3}{9\pi}}\left[\frac{\cos\alpha(1-\sin\alpha)^{\frac{3}{2}}\sec^2\alpha + 2\sqrt{\sin\alpha}(1-\sin\alpha)^{\frac{3}{2}}\sec^2\alpha\tan\alpha}{2\sqrt{\sin\alpha}} - \frac{3\sqrt{\sin\alpha}\sec^2\alpha(1-\sin\alpha)^{\frac{1}{2}} \times \cos\alpha}{2}\right] = 0$$

$\therefore$ $1 - \sin\alpha = 0$

$\Rightarrow$ $\sin\alpha = 1$ rejected as $\alpha \in \left(0, \frac{\pi}{2}\right)$

OR

$$\therefore \quad \frac{\cos\alpha(1-\sin\alpha)\sec^2\alpha + 2\sqrt{\sin\alpha}(1-\sin\alpha)\sec^2\alpha\tan\alpha}{2\sqrt{\sin\alpha}}$$

$$= 3\frac{\sqrt{\sin\alpha}\sec^2\alpha\cos\alpha}{2}$$

$$\therefore \quad \frac{\cos\alpha(1-\sin\alpha) + 2\sin\alpha(1-\sin\alpha)\tan\alpha}{2} = \frac{3\sin\alpha\cos\alpha}{2}$$

$$\therefore \quad \cos\alpha + 4\sin\alpha\tan\alpha - 4\sin^2\alpha\tan\alpha = 4\sin\alpha\cos\alpha$$

$\therefore \quad \cos^2\alpha + 4\sin^2\alpha - 4\sin^3\alpha = 4\sin\alpha\left(1 - \sin^2\alpha\right)$

$\therefore \quad 3 + 3\sin^2\alpha - 4\sin^3\alpha = 4\sin\alpha - 4\sin^3\alpha$

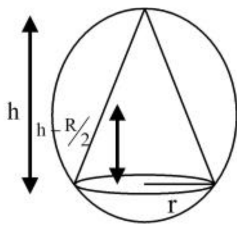
$\therefore \quad 3\sin^2\alpha - 4\sin\alpha + 3 = 0$

$\Rightarrow \quad \sin\alpha = \frac{1}{3}$ hence proved.

20. Refer (Q.21) Assifnment

Ans. $\frac{1}{3}$m

21.



$r^2 + (h - R)^2 = R^2$

$\therefore \quad r^2 = 2Rh - h^2$

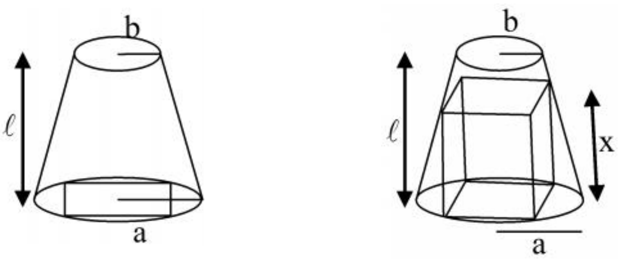
$\therefore \quad V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(2Rh^2 - h^3\right)$

$\therefore \quad \frac{dV}{dh} = \frac{1}{3}\left(4Rh - 3h^3\right) = 0$

$\therefore \quad h = 0$ Rejected or $h = \frac{4R}{3}$

22.



Where h is the height of cut off conical portion

$$\frac{b}{h} = \frac{a}{\ell + h} = \frac{s}{\sqrt{2}\left(h + \ell - x\right)}$$

$$\therefore \quad s = \frac{\sqrt{2}\, b\left(h + \ell - x\right)}{h}$$

$$V = \frac{2b^2\left(h + \ell - x\right)^2}{h^2} \times x$$

$$= \frac{2b^2}{h^2}\left(h^2x + \ell^2x + x^3 + 2h\ell x - 2hx^2 - 2\ell x^2\right)$$

$\therefore \quad \frac{dV}{dx} = \frac{2b^2}{h^2}\left(h^2 + \ell^2 + 2h\ell + 3x^2 - 4hx - 4\ell x\right) = 0$

$\therefore \quad 3x^2 - 4(h-\ell)x + (h+\ell)^2 = 0$

$\therefore \quad x = h + \ell$ rejected or

$x = \frac{(h+\ell)}{3}$

Now, $b\ell + bh = ah$

$\therefore \quad h = \frac{b\ell}{a-b}$

$\therefore \quad x = \frac{1}{3}\left(\frac{b\ell}{a-b} + \ell\right) = \frac{a\ell}{3(a-b)}$

23. Profit = SP − CP

$$= \frac{n}{2}(100-n) - \frac{n^2}{4} - 35n - 25$$

$P = -\frac{3n^2}{4} + 15n - 25$

$\therefore \quad \frac{dP}{dn} = -\frac{3n}{2} + 15 = 0$

$\therefore \quad n = 10$

24. $R = (500-x)(300+x)$ $\qquad$ x = Surcharge

$\therefore \quad \frac{dR}{dx} = -300 - x + 500 - x = 0$

$\therefore \quad x = 100$

25. $f(x) = |2x-1| - 2|x-1| - 3$

$\therefore \quad f(x) = 1 - 2x + 2x - 2 - 3 = -4, \quad x \le \frac{1}{2}$

$$= 2x - 1 + 2x - 2 - 3 = 4x - 6, \; \frac{1}{2} < x \le 1$$

$$= 2x - 1 - 2x + 2 - 3 = -2, \quad x > 1$$

$\therefore$ Range of $f(x)$ is $[-4, -2]$

26. $x + y = 20$ $\qquad$ $y = 20 - x$

$x^3y^2 = \max = P$

$\therefore \quad P = x^3(20-x)^2$

$$= x^5 - 40x^4 + 400x^3$$

$\therefore \quad \frac{dP}{dx} = 5x^4 - 160x^3 + 1200x^2 = 0$

$\therefore \quad x = 0 \quad \Rightarrow$ Rejected or

$5x^2 - 160x + 1200 = 0$

$\Rightarrow \quad x^2 - 32x + 240 = 0$

$\therefore \quad x = \frac{32 \pm \sqrt{64}}{2} = 20 \text{ or } 12$

$x = 20$ rejected

$\therefore \quad x = 12 \;\&\; y = 8$

27. $f(a)g(b) - f(b)g(a) = (b-a)[f(a)g'(c) - g(a)f'(c)]$

29. $f(x) = x(x+3)e^{-x/2} \qquad [-3, 0]$

$f'(x) = (2x+3)e^{-x/2} - \frac{1}{2}x(x+3)e^{-x/2} = 0$

at $x = c$

$\therefore \quad 2(2x+3) = x^2 + 3x$

$\therefore \quad x^2 - x - 6 = 0$

$\therefore \quad x = -2$ i.e. as $c < 0$

i.e. $\quad c = -2$

30. $f(x) = x(x-1)(x-2)$

$\therefore \quad f(0) = 0, f\left(\frac{1}{2}\right) = \frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2} = \frac{3}{8}$

$\therefore \quad f'(x) = (x-1)(x-2) + x(x-2) + x(x-1)$

$f'(c) = \frac{\frac{3}{8}}{\frac{1}{2}} = \frac{3}{4}$

$\therefore \quad \frac{3}{4} = 3c^2 - 6c + 2$

$\therefore \quad 12c^2 - 24c + 5 = 0$

$\therefore \quad c = \frac{24 \pm \sqrt{576 - 240}}{24} = \frac{24 \pm \sqrt{336}}{24} = \frac{6 - \sqrt{21}}{6}$ as $c < \frac{1}{2}$