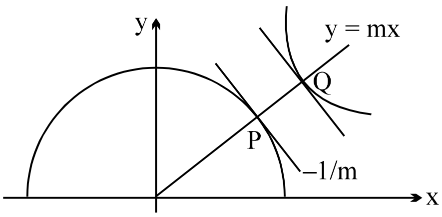
Applications Of Derivatives
Exercise 2(B)

1 Given $S = x^2 + 4xh = 1200$
and $V = x^2h$

$V(x) = \dfrac{x^2(1200 - x^2)}{4x}$; $V(x) = \dfrac{1}{4}(1200x - x^3)$

Put $V'(x) = 0$ gives $x = 20$
If $x = 20, h = 10$
Hence $V_{max.} = x^2h = (400)(10) = 4000$ cubic cm.

2 Note that C_1 is a semicircle and C_2 is a rectangular hyperbola.
PQ will be minimum if the normal at P on the semicircle is also a normal at Q on $xy = 9$
Let the normal at P be $y = mx$ ....(1) $(m > 0)$
solving it with $xy = 9$



$mx^2 = 9 \Rightarrow x = \dfrac{3}{\sqrt{m}}$; $y = \dfrac{9\sqrt{m}}{3}$

$\therefore$ $Q \equiv \left(\dfrac{3}{\sqrt{3}}, 3\sqrt{m}\right)$

differentiating $xy = 9$

$x\dfrac{dy}{dx} + y = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{y}{x}$

$\therefore$ $\left.\dfrac{dy}{dx}\right|_Q = -\dfrac{3\sqrt{m}\cdot\sqrt{m}}{3} = -m$

$\therefore$ tangent at P and Q must be parallel

$\therefore$ $-m = -\dfrac{1}{m} \Rightarrow m^2 = 1 \Rightarrow m = 1$

$\therefore$ normal at P and Q is $y = x$
solving P(1, 1) and Q(3, 3)
$\therefore$ $(PQ)^2 = d^2 = 4 + 4 = 8$ Ans.]

3 The given expression resembles with $(x_1 - x_2)^2 + (y_1 - y_2)^2$, where $y_1 = \dfrac{x_1^2}{20}$ and

$y_2 = \sqrt{(17 - x_2)(x_2 - 13)}$

Thus, we can thing about two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lying on the curves $x^2 = 20y$ and $(x - 15)^2 + y^2 = 4$ respectively.
Let D be the distance between P_1 and P_2 then the given expression simply represents D^2.
Now, as per the requirements, we have to locate the point on these curves (in the first quadrant) such that the distance between them is minimum.
Since the shortest distance between two curves always occurs along the common normal, it implies that we have to locate a point $P(x_1, y_1)$ on the parabola $x^2 = 20y$ such that normal drawn to parabola at this point passes through (15, 0).

Now, equation of the normal to the parabola at (x_1, y_1) is $\left(y - \dfrac{x_1^2}{20}\right) = \dfrac{-10}{x_1}(x - x_1)$. It should pass through (15, 0).

$\Rightarrow$ $x_1^3 + 200x_1 - 3000 = 0$ $\Rightarrow$ $x_1 = 10 \Rightarrow$ $y_1 = 5$

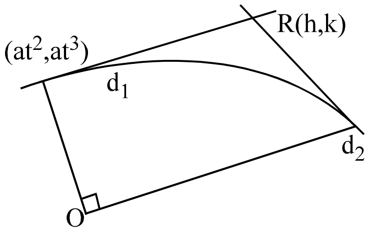
$\Rightarrow \quad D = \sqrt{(10-15)^2 + 5^2} - 2 = \left(5\sqrt{2} - 2\right)$

The minimum value of the given expression is $\left(5\sqrt{2}-2\right)^2 = \left(a\sqrt{2} - b\right)^2$

$\therefore \quad a = 5$ & $b = 2$

4 $\quad x = t^2$; $y = t^3$

$\frac{dx}{dt} = 2t$; $\frac{dy}{dt} = 3t^2$



$\frac{dy}{dx} = \frac{3t}{2}$

$y - t^3 = \frac{3t}{2}(x - t^2) \quad ....(1)$

$2k - 2t^3 = 3th - 3t^3$

$t^3 - 3th + 2k = 0$

$t_1 t_2 t_3 = -2k \quad$ (put $t_1 t_2 = -1$); $\quad$ hence $t_3 = 2k$

now t_3 must satisfy the equation (1) which gives $4y^2 = 3x - 1$.
Comparing with $ay^2 = bx - 1$, we have a = 4 and b = 3.

5 $\quad$ We have $F(x) = \begin{cases} -2x + \log_{\frac{1}{2}}(k^2 - 6k + 8), & -2 \le x < -1 \\ x^3 + 3x^2 + 4x + 1, & -1 \le x \le 3 \end{cases}$

Also F(x) is increasing on [–1, 3] because $F'(x) > 0 \ \forall \ x \in [-1, 3]$.
And $F'(x) = -2 \ \forall \ x \in [-2, -1)$, so F(x) is decreasing on [–2,–1).

$\therefore$ If F(x) has smallest value at x = –1, then we must have

$\lim_{h \to 0} F(-1-h) \ge F(-1)$

$\Rightarrow \quad 2 + \log_{1/2}(k^2 - 6k + 8) \ge -1 \quad \Rightarrow \quad \log_{1/2}(k^2 - 6k + 8) \ge -3 \quad \Rightarrow \quad k^2 - 6k + 8 \le 8$

$\Rightarrow \quad k^2 - 6k \le 0 \quad \Rightarrow \quad k \in [0, 6] \quad ....(1)$

But in order to define $\log_{1/2}(k^2 - 6k + 8)$,

We must have $k^2 - 6k + 8 > 0$

$\Rightarrow \quad (k-2)(k-4) > 0 \quad \Rightarrow \quad k < 2$ or $k > 4 \quad ......(2)$

$\therefore$ From (1) and (2), we get $k \in [0, 2) \cup (4, 6]$

$\Rightarrow$ Possible integer(s) in the range of k are 0, 1, 5, 6

Hence the sum of all possible positive integer(s) in the range of k = 1 + 5 + 6 = 12 **Ans.**]

6 $\quad$ We have $F(x) = \frac{x^3}{3} + (a-3)x^2 + x - 13$

$\therefore$ For F(x) to have negative point of local minimum, the equation F '(x) = 0 must have two distinct negative roots.

Now, $\quad F'(x) = x^2 + 2(a-3)x + 1$

$\therefore$ Following condition(s) must be satisfied simultaneously.

(i) Discriminant > 0; $\quad$ (ii) Sum of roots < 0 ; $\quad$ (iii) Product of roots > 0

Now, $\quad D > 0$

$\Rightarrow \quad 4(a-3)^2 > 4 \quad \Rightarrow \quad (a-3)^2 - 1 > 0 \quad \Rightarrow \quad (a-2)(a-4) > 0$

$\therefore \quad a \in (-\infty, 2) \cup (4, \infty) \quad .... (i)$

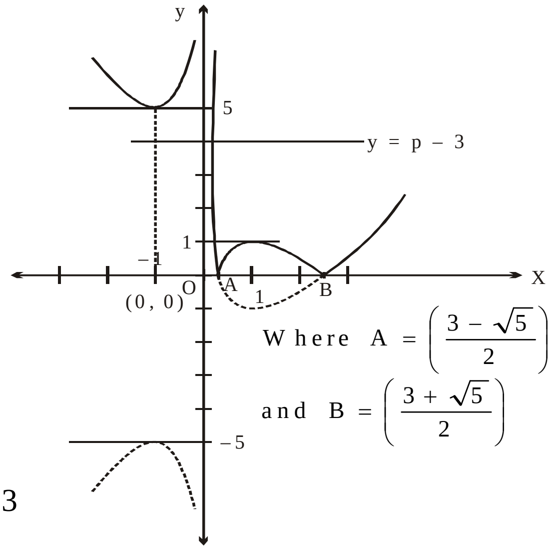
Also $\quad -2(a-3) < 0 \quad \Rightarrow \quad a - 3 > 0 \quad \Rightarrow \quad a > 3 \quad .... (ii)$

And product of root(s) = $1 > 0 \ \forall \ a \in R$

$\therefore \quad (i) \cap (ii) \cap (iii) \quad \Rightarrow \quad a \in (4, \infty) \quad ....(iii)$

Hence sum of value(s) of a = 5 + 6 + 7 + ... + 100 = 5040 $\quad$ Ans.]

7. Consider $y = x + \frac{1}{x} - 3$



$\Rightarrow \quad \frac{dy}{dx} = 1 - \frac{1}{x^2} = 0$

$\therefore \quad \frac{dy}{dx} = 0 \quad \Rightarrow \quad x = 1$ or -1

As $x \to 0^+$, $y \to \infty$ and $x \to 0^-$, $y \to -\infty$

Also roots of $x + \frac{1}{x} - 3 = 0 \Rightarrow x^2 - 3x + 1 = 0$

$$x = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

For two distinct solutions either $p - 3 = 0 \Rightarrow p = 3$

or $1 < p - 3 < 5$

$4 < p < 8$

Hence $p \in \{3\} \cup (4, 8)$

$p = \{3, 5, 6, 7\} \quad \Rightarrow \quad$ Sum = 21 **Ans.**]

8 Let $f''(x) = 6a(x-1)$ $(a > 0)$ then $f'(x) = 6a\left(\frac{x^2}{2} - x\right) + b = 3a(x^2 - 2x) + b$.

Now $f'(-1) = 0 \Rightarrow 9a + b = 0 \Rightarrow b = -9a$.

$\therefore \quad f'(x) = 3a(x^2 - 2x - 3) = 0 \Rightarrow x = -1$ and 3.

So $y = f(-1)$ and $y = f(3)$ are two horizontal tangents.

Hence distance between its two horizontal tangents = $|f(3) - f(-1)| = |-22 - 10| = 0032$. **Ans.**]

9 Volume (V) = $\frac{1}{3} A_1 h_1 \quad \Rightarrow h_1 = \frac{3V}{A_1}$

|||ly $\quad h_2 = \frac{3V}{A_2}$, $h_3 = \frac{3V}{A_3}$ and $h_4 = \frac{3V}{A_4}$

So $(A_1 + A_2 + A_3 + A_4)(h_1 + h_2 + h_3 + h_4) = (A_1 + A_2 + A_3 + A_4)\left(\frac{3V}{A_1} + \frac{3V}{A_2} + \frac{3V}{A_3} + \frac{3V}{A_4}\right)$

$$= 3V(A_1 + A_2 + A_3 + A_4)\left(\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4}\right)$$

Now using A.M.-H.M inequality in A_1, A_2, A_3, A_4, we get

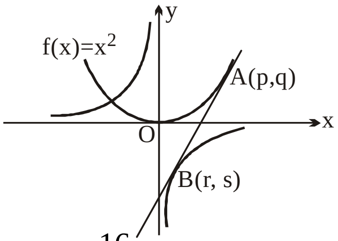
$$\frac{A_1 + A_2 + A_3 + A_4}{4} \geq \frac{4}{\left(\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4}\right)}$$

$$\Rightarrow (A_1+A_2+A_3+A_4)\left(\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4}\right) \geq 16$$

Hence the minimum value of $(A_1+A_2+A_3+A_4)(h_1+h_2+h_3+h_4) = 3V(16) = 48V = 48 \times 5 = 240$ **Ans.**]

10 $y = x^2$ and $y = -\frac{8}{x}$; $q = p^2$ and $s = -\frac{8}{r}$ ....(1)

Equating $\frac{dy}{dx}$ at A and B, we get



$2p = \frac{8}{r^2}$ ....(1) $\Rightarrow$ $pr^2 = 4$

Now $m_{AB} = \frac{q-s}{p-r}$ $\Rightarrow$ $2p = \frac{p^2 + \frac{8}{r}}{p-r}$ $\Rightarrow$ $p^2 = 2pr + \frac{8}{r}$ $\Rightarrow$ $p^2 = \frac{16}{r}$

$\Rightarrow$ $\frac{16}{r^4} = \frac{16}{r}$ $\Rightarrow$ $r = 1$ $(r \neq 0)$ $\Rightarrow$ $p = 4$

$\therefore$ $r = 1,\ p = 1$

Hence $p + r = 5$

11 $x = 0$ and $x = 1$]

12 $y = x^2$

$\frac{dy}{dt} = 2x \cdot \frac{dx}{dt}$

$\frac{dx}{dt} = 10$m/sec.

$\tan\theta = \frac{x^2}{x} = x$

$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{dx}{dt}$

$\frac{d\theta}{dt} = 10 \times \cos^2\theta = 10 \times \frac{1}{10} = 1$ { at x = 3m }

13 $3x^2 - 2y\frac{dy}{dx} = 0$ $\Rightarrow$ $\frac{dy}{dx} = \frac{3x^2}{2y}$

slope of tangent at $(4a^2, 8a^3) = \frac{3(16a^4)}{2(8a^3)} = 3a$

let this tangent at this point also cuts the curve at $(4b^2, 8b^3)$ and normal at this point slope of

normal at $(4b^2, 8b^3) = -\frac{1}{3b}$.

$\Rightarrow$ $3a = -\frac{1}{3b}$ $\Rightarrow$ $ab = -\frac{1}{9}$ ......(i)

slope of line $= \frac{8a^3 - 8b^3}{4a^2 - 4b^2} = \frac{2(a^3 - b^3)}{(a^2 - b^2)} = \frac{2(a^2 + b^2 + ab)}{a+b}$

$= 3a$ [it is equal to slope of target]

$\Rightarrow$ $2a^2 + 2b^2 + 2ab = 3a^2 + 3ab$

$\Rightarrow\ 2b^2 = a^2 + ab \quad \Rightarrow\ \dfrac{2}{81a^2} = \dfrac{a^2-1}{9}$

$2 = 81a^4 - 9a^2$

$\Rightarrow\ 81a^4 - 9a^2 - 2 = 0$

$81a^4 - 18a^2 + 9a^2 - 2 = 0$

$9a^2(9a^2 - 2) + (9a^2 - 2) = 0$

$\Rightarrow\ (9a^2 - 2)(9a^2 + 1) = 0$

$9a^2 = 2$

14. Let $x = r\cos\theta$ and $y = r\sin\theta$

$\Rightarrow\quad r^2 = x^2 + y^2;\quad \tan\theta = \dfrac{y}{x} \qquad \theta \in (0, \pi/2)$

$$N = \frac{r^2}{r^2[\cos^2\theta + \sin\theta\cos\theta + 4\sin^2\theta]} = \frac{r^2}{(1+\cos 2\theta) + \sin 2\theta + 4(1-\cos 2\theta)} = \frac{2}{5 + \sin 2\theta + 3\cos 2\theta}$$

$$N_{max} = \frac{2}{5-\sqrt{10}} = \frac{2}{15}\left(5+\sqrt{10}\right) = M$$

$$N_{max} = \frac{2}{5+\sqrt{10}} = \frac{2}{15}\left(5-\sqrt{10}\right) = m$$

$$A = \frac{M+m}{2} = \frac{2\cdot 10}{15\cdot 2} = \frac{2}{3} \qquad \Rightarrow \qquad 2007 \times \frac{2}{3} = 1338 \text{ Ans.]}$$

15. $\dfrac{f(3)}{f(6)} = \dfrac{2^{3k}+9}{2^{6k}+9} = \dfrac{1}{3};\qquad f(9) - f(3) = (2^{9k} + 9) - (2^{3k} + 9) = 2^{9k} - 2^{3k}$ ....(1)

$3(2^{3k} + 9) = 2^{6k} + 9$

$\Rightarrow\quad 2^{6k} - 3(2^{3k}) - 18 = 0$

$2^{3k} = y$

$y^2 - 3y - 18 = 0$

$(y - 6)(y + 3) = 0$

$y = 6;\quad y = -3$ (rejected)

$2^{3k} = 6$

now $f(9) - f(3) = 2^{9k} - 2^{3k}$ { from (1) }

$= (2^{3k})^3 - 2^{3k}$

$= 6^3 - 6 = 210$

hence $N = 210 = 2\cdot 3\cdot 5\cdot 7$

Total number of divisor $= 2\cdot 2\cdot 2\cdot 2 = 16$

number of divisors which are composite = 16 – (1, 2, 3, 5, 7) = 11 Ans.]

16. $f(-3) = f(3) = 2$ [$f(x)$ is an even function, $\therefore\ f(-x) = f(x)$]

again $f(-1) = f(1) = -3$

$\therefore\quad 2|f(-1)| = 2|f(1)| = 2|-3| = 6$

from the graph, $\quad -3 < f\left(\dfrac{7}{8}\right) < -2$

$\therefore\quad \left[f\left(\dfrac{7}{8}\right)\right] = -3$

$f(0) = 0$ (obviously from the graph)

$\cos^{-1}(f(-2)) = \cos^{-1}(f(2)) = \cos^{-1}(1) = 0$

$f(-7) = f(-7+8) = f(1) = -3$ [f (x) has period 8]

$f(20) = f(4+16) = f(4) = 3$ [$f(nT + x) = f(x)$]

sum = 2 + 6 – 3 + 0 + 0 – 3 + 3

$\therefore$ sum = 5

17 We have $f(x) = (b^2 - 3b + 2)(\cos^2 x - \sin^2 x) + (b-1)x + \sin 2$

$\therefore$ $f'(x) = (b-1)(b-2)(-2\sin 2x) + (b-1)$

Now, $f'(x) \neq 0$ for every $x \in R$,

so $(b-1)\left(1 - 2(b-2)\ \sin 2x\right) \neq 0\ \forall\ x \in R$

$\therefore$ $b \neq 1$

Also, $\left|\dfrac{1}{2(b-2)}\right| > 1 \Rightarrow b \in \left(\dfrac{3}{2}, 2\right) \cup \left(2, \dfrac{5}{2}\right)$

Now, when b = 2, $f(x) = x + \sin 2 \Rightarrow f'(x) = 1\ (\neq 0)$.

Hence, $b \in \left(\dfrac{3}{2}, \dfrac{5}{2}\right) \Rightarrow b_1 = \dfrac{3}{2}$ and $b_2 = \dfrac{5}{2}$

$\Rightarrow (b_1 + b_2) = \dfrac{3}{2} + \dfrac{5}{2} = \dfrac{8}{2} = 4$

18. Let $x = r\cos\theta$ and $y = r\sin\theta$

$E = (x+5)(y+5) = (r\cos\theta + 5)(r\sin\theta + 5) = r^2 \sin\theta\cos\theta + 5r(\cos\theta + \sin\theta) + 25$

Now put $x = r\cos\theta$ and $y = r\sin\theta$ in $x^2 + xy + y^2 = 3$

$\Rightarrow r^2\cos^2\theta + r^2\sin\theta\cos\theta + r^2\sin^2\theta = 3$

$\Rightarrow r^2(1 + \sin\theta\cos\theta) = 3 \Rightarrow r^2 = \dfrac{3}{1 + \sin\theta\cos\theta} = \dfrac{6}{2 + \sin 2\theta}$

hence $r^2]_{min.} = 2 + \sin 2\theta]_{max.} = 3$ occurs at $\sin 2\theta = 1 \Rightarrow 2\theta = \dfrac{\pi}{2}$ or $\dfrac{5\pi}{2}$ i.e. $\dfrac{\pi}{4}$ or $\dfrac{5\pi}{4}$

Hence $E = \dfrac{r^2}{2}(\sin 2\theta) + 5r(\cos\theta + \sin\theta) + 25$

put $r^2 = 2$ and $\theta = \dfrac{\pi}{4} \Rightarrow E = 1 + 5\sqrt{2}\left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}\right) + 25 = 36$

put $r^2 = 2$ and $\theta = \dfrac{5\pi}{4} \Rightarrow E = 1 + 5\sqrt{2}\left(-\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}\right) + 25 = 16$

hence minimum value of E is 16

19. Using LMVT for f in [1, 2]

$\forall\ c \in (1, 2)\quad \dfrac{f(2) - f(1)}{2-1} = f'(c) \leq 2$

$f(2) - f(1) \leq 2 \Rightarrow f(2) \leq 4$ ....(1)

again using LMVT in [2, 4]

$\forall\ d \in (2, 4)\quad \dfrac{f(4) - f(2)}{4-2} = f'(d) \leq 2$

$\therefore$ $f(4) - f(2) \leq 4$

$8 - f(2) \leq 4$

$4 \leq f(2) \quad \Rightarrow \quad f(2) \geq 4 \quad ....(2)$

from (1) and (2) $\quad f(2) = 4$

20. Let x tree be added then

$P(x) = (x + 50)(800 - 10x)$

now $\quad P'(x) = 0 \quad \Rightarrow \quad x = 15$