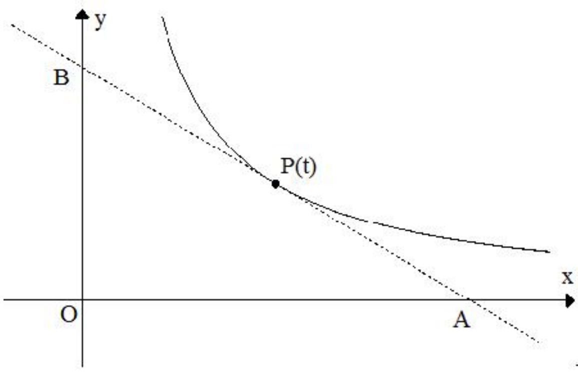
APPLICATIONS OF DERIVATIVES
EXERCISE 1(C)

1. Let P be $\left(ct, \frac{c}{t}\right)$, then



$xy = c^2 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$ or $\frac{dy}{dx} = -\frac{1}{t^2}$

tangent at P will be

$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$ or $x + t^2 y = 2ct$

Now $OA = |2ct|, OB = \left|\frac{2c}{t}\right|$

$\Delta = \frac{1}{2}(2ct)\left(\frac{2c}{t}\right) = 2c^2$

2. $x^2 = 4y$

$\therefore \quad \frac{dy}{dx} = \frac{x}{2}$

$\therefore \quad -\frac{dx}{dy} = -\frac{2}{x} = -2$ at $(1, 2)$

FInd equation of line passing through (1,2) with slope -2.

3. $\frac{dy}{dx} = \frac{a\theta \sin\theta}{a\theta \cos\theta} = \tan\theta$

$\therefore \quad -\frac{dx}{dy} = -\cot\theta$

$\therefore \quad \frac{Y - a(\sin\theta - \theta\cos\theta)}{X - a(\cos\theta + \theta\sin\theta)} = -\cot\theta$

$\therefore \quad Y - a\sin\theta + a\theta\cos\theta = -\cot\theta X + a\cos\theta\cot\theta + a\theta\cos\theta$

$\therefore \quad Y + X\cot\theta - a(\sin\theta + \cos\theta\cot\theta) = 0$

$\therefore \quad$ Distance from origin $= \frac{|a(\sin\theta + \cos\theta\cot\theta)|}{\sqrt{1 + \cot^2\theta}}$

$= a$

4. $\frac{dy}{dx} = \frac{2ae^{\theta}\cos\theta}{2ae^{\theta}\sin\theta} = \cot\theta$

$-\frac{dy}{dx} = -\tan\theta$

Equation of tangent is

$$\frac{y-ae^{\theta}(\sin\theta+\cos\theta)}{x-ae^{\theta}(\sin\theta-\cos\theta)}=\cot\theta$$

$\therefore \quad y-ae^{\theta}\sin\theta-ae^{\theta}\cos\theta=x\cot\theta-ae^{\theta}\cos\theta+ae^{\theta}\cos\theta\cot\theta$

$\therefore \quad \frac{x\cos\theta}{\sin\theta}-y+ae^{\theta}\sin\theta+ae^{\theta}\frac{\cos^2\theta}{\sin\theta}=0$

$\therefore \quad x\cos\theta-y\sin\theta+ae^{\theta}=0$

$\therefore \quad p=\frac{|ae^{\theta}|}{1}=ae^{\theta}$

Equation of normal is

$$\frac{y-ae^{\theta}(\sin\theta+\cos\theta)}{x-ae^{\theta}(\sin\theta-\cos\theta)}=-\tan\theta$$

$\therefore \quad y-ae^{\theta}\sin\theta-ae^{\theta}\cos\theta=-x\tan\theta+ae^{\theta}\sin\theta\tan\theta-ae^{\theta}\sin\theta$

$\therefore \quad y\cos\theta+x\sin\theta-ae^{\theta}=0$

$\therefore \quad q=\frac{|-ae^{\theta}|}{1}=ae^{\theta}$

$\therefore \quad p=q$

5. $x^{2/3}+y^{2/3}=a^{2/3}$

$\therefore \quad \frac{2}{3x^{1/3}}+\frac{2}{3y^{1/3}}\frac{dy}{dx}=0$

$\therefore \quad \frac{dy}{dx}=-\left(\frac{y}{x}\right)^{1/3}$

$\therefore \quad -\frac{dx}{dy}=\left(\frac{x}{y}\right)^{1/3}$

Equation of tangent is

$$\frac{Y-y}{X-x}=-\left(\frac{y}{x}\right)^{1/3}$$

$\therefore \quad x^{1/3}Y-x^{1/3}y=-y^{1/3}X+xy^{1/3}$

$\therefore \quad y^{1/3}X+x^{1/3}Y-x^{1/3}y-xy^{1/3}=0$

$$p=\frac{|x^{1/3}y+xy^{1/3}|}{\sqrt{x^{2/3}+y^{2/3}}}$$

$$=|x^{1/3}y^{1/3}a^{1/3}|$$

Equation of normal is

$$\frac{Y-y}{X-x}=\left(\frac{x}{y}\right)^{1/3}$$

$\therefore \quad y^{1/3}Y-y^{4/3}=x^{1/3}X-x^{4/3}$

$\therefore \quad x^{1/3}X-y^{1/3}Y-x^{4/3}+y^{4/3}=0$

$$\therefore \quad q=\frac{\left|y^{4/3}-x^{4/3}\right|}{\sqrt{x^{2/3}+y^{2/3}}}$$

$$=\left|\left(x^{2/3}-y^{2/3}\right)a^{1/3}\right|$$

$\therefore \quad 4p^2+q^2=4x^{2/3}y^{2/3}+a^{2/3}\left(x^{4/3}-2x^{2/3}y^{2/3}+y^{4/3}\right)$

$$=a^{2/3}\left(x^{2/3}+y^{2/3}\right)$$

$$=a^2.$$

6. $y^2=2x, x^2+y^2=8$

$\therefore \quad x^2+2x-8=0$ and $x>0$ as $x=\frac{y^2}{2}$

$\therefore \quad x=2$

$\Rightarrow \quad y=2$

For $y^2=2x$

$\therefore \quad 2y\frac{dy}{dx}=2$

$\therefore \quad m_1=\frac{1}{y}=\frac{1}{2}$

For $x^2+y^2=8$

$\therefore \quad 2x+2y\frac{dy}{dx}=0$

$\therefore \quad m_2=-\frac{x}{y}=-1$

$\therefore \quad \tan\alpha=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=3$

7. $y=\frac{x+3}{x^2+1}$

$$\therefore \quad \frac{dy}{dx}=\frac{\left(x^2+1\right)-\left(2x^2+6x\right)}{\left(x^2+1\right)^2}=\frac{-x^2-6x+1}{\left(x^2+1\right)^2}$$

$\therefore \quad m_1 = \frac{-4-12+1}{25} = \frac{-3}{5}$

$y = \frac{x^2-7x+11}{x-1}$

$\therefore \quad \frac{dy}{dx} = \frac{\left(2x^2-9x+7\right)-\left(x^2-7x+11\right)}{\left(x-1\right)^2} = \frac{x^2-2x-4}{\left(x-1\right)^2}$

$\therefore \quad m_2 = \frac{4-4-4}{1} = -4$

$\therefore \quad \tan\alpha = \left|\frac{m_1-m_2}{1+m_1m_2}\right| = 4$

8. $x^3-3xy^2+2=0$

$\therefore \quad 3x^2-3xy^2-6xy\frac{dy}{dx}=0$

$\therefore \quad x^2-y^2 = 2xy\frac{dy}{dx}$

$\therefore \quad \frac{dy}{dx} = \frac{x^2-y^2}{2xy} = m_1$

$3x^2y-y^3+2=0$

$\therefore \quad 6xy+3x^2\frac{dy}{dx}-3y^2\frac{dy}{dx}=0$

$\therefore \quad \frac{dy}{dx}\left(x^2-y^2\right) = -2xy$

$\therefore \quad \frac{dy}{dx} = \frac{-2xy}{x^2-y^2} = m_2$

$m_1m_2 = -1$

9. $x = y^2$

$\therefore \quad \frac{dy}{dx} = \frac{1}{2y} = m_1$

$xy = k$

$\therefore \quad \frac{dy}{dx} = -\frac{y}{x} = m_2$

$m_1m_2 = -1$

$\therefore \quad \frac{-1}{2x} = -1$

$\therefore \quad x = \frac{1}{2}$

$\therefore \quad y = \pm\frac{1}{\sqrt{2}}$

$\therefore \quad k = \pm\frac{1}{2\sqrt{2}}$

10. $ST = \frac{3}{8}, SN = 24$

$y_0^2 = ST.SN$

$= \frac{3}{8} \times 24 = 9$

$\therefore \quad y_0 = \pm 3$

11. $by^2 = (x+a)^3$

$\therefore \quad 2by\frac{dy}{dx} = 3(x+a)^2$

$\therefore \quad \frac{dy}{dx} = \frac{3(x+a)^2}{2by} = \tan\theta$

$\cot\theta = \frac{2by}{3(x+a)^2}$

$ST = |y\cot\theta| = \left|\frac{2by^2}{3(x+a)^2}\right|$

$SN = |y\tan\theta| = \left|\frac{3(x+a)^2}{2b}\right|$

$\therefore \quad \frac{3p(x+a)^2}{2b} = \frac{4qb^2y^4}{9(x+a)^4}$

$\therefore \quad \frac{p}{q} = \frac{8b^3y^4}{27(x+a)^6} = \frac{8b}{27}\frac{(by^2)^2}{(x+a)^6} = \frac{8b}{27}$

12. $xy^n = a^{n+1}$

$\therefore \quad \frac{dy}{dx} = \frac{-y}{nx} = \tan\theta$

$\therefore \quad SN = |y\tan\theta|$

$= \left|\frac{-y^2}{nx}\right| = \text{constant}$

But xy^n = constant

$\Rightarrow \quad n = -2$

13. $x^2y^2 = a^5$

$\therefore \quad 2xy^2 + 2x^2y\dfrac{dy}{dx} = 0$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{-y}{x} = \tan\theta$

$\therefore \quad \cot\theta = \dfrac{-x}{y}$

$\therefore \quad ST = |y\cot\theta| = |-x|$

14. $x^m y^n = a^{m+n}$

$\therefore \quad mx^{m-1}y^n + nx^m y^{n-1}\dfrac{dy}{dx} = 0$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{my}{nx}$

$\therefore \quad \cot\theta = -\dfrac{nx}{my}$

$\therefore \quad ST = \left|-\dfrac{nx}{m}\right|$

15. From informationin Q.22,

$(ST)^2 \propto (SN)$

16. $-\dfrac{dx}{dy} = -\dfrac{a\theta\cos\theta}{a\theta\sin\theta} = -\cot\theta$

As shown in Q.14 of this exercise it is at a constant distance from origin.

17. $ax + by + c = 0$ normal to $xy = 1$

For $xy = 1, \dfrac{dy}{dx} = -\dfrac{y}{x}$

As xy is positive, $\dfrac{dy}{dx} < 0 \quad \forall\, x, y$

$\therefore \quad -\dfrac{dx}{dy} < 0 \quad \forall\, x, y$

$\therefore$ Slope of normal is positive.

$\therefore \quad a > 0, b < 0$ or $a < 0, b > 0$

18. $(3-a)x + ay + a^2 - 1 = 0$

$\therefore \quad -\left(\frac{a}{3-a}\right) > 0$

$\therefore \quad a \in (-\infty, 0) \cup (3, \infty)$

19. $f(x) = 2x^2 - \log|x|$

$\therefore \quad f'(x) = 4x - \frac{1}{x} < 0$

$\therefore \quad \frac{4x^2 - 1}{x} < 0$

$\therefore \quad \frac{(2x+1)(2x-1)}{x} < 0 \qquad \therefore \quad x \in \left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

20. $f(x) = \frac{x}{\sin x}$, $g(x) = \frac{x}{\tan x}$

$f'(x) = \frac{\sin x - x\cos x}{x^2} > 0 \quad \forall\, x \in (0, 1)$

$g'(x) = \frac{\tan x - x\sec^2 x}{x^2} < 0 \quad \forall\, x \in (0, 1)$

21. $f(x) = \tan^{-1}(\sin x + \cos x)$

Let $\quad g(x) = \tan^{-1} x$

$\therefore \quad g'(x) = \frac{1}{1+x^2} > 0 \quad \forall\, x$

$\therefore \quad f(x)$ increases when $\sin x + \cos x$ increases

Let $\quad h(x) = \sin x + \cos x$

$\therefore \quad h'(x) = \cos x - \sin x > 0$

$\therefore \quad \cos x > \sin x$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \quad x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

22. $f(x) = x^{100} + \sin x - 1$

$f'(x) = 100x^{99} + \cos x < 0$

23. $f(x) = |x| - |x-1|$

$x < 0 \quad \Rightarrow \quad f(x) = -x + 1 - x = 1 - 2x$ MD

$0 < x < 1 \quad \Rightarrow \quad f(x) = x + 1 - x = 1$ Constant

$x > 1 \quad \Rightarrow \quad f(x) = 2x - 1$ MI

24. $f(x) = x(a^2 - 2a - 2) + \cos x$

$f'(x) = a^2 - 2a - 2 - \sin x > 0 \quad \forall x$

$\therefore \quad a^2 - 2a - 2 > 1$

$\therefore \quad a \in (-\infty, -1) \cup (3, \infty)$

25. $\phi(x) = 3f\left(\frac{x^3}{3}\right) + f(3 - x^2) \;\; \forall \;\; x \in (-3, 4)$

$f''(x) > 0$

$\therefore \quad \phi'(x) = 3x^2 f'\left(\frac{x^3}{3}\right) - 2x f'(3 - x^2) > 0$

$x^2 f'\left(\frac{x^3}{3}\right) > 2x \; f'(3 - x^2)$

For $x > 0$

$x f'\left(\frac{x^3}{3}\right) > 2 f'(3 - x^2)$

$\therefore \quad \frac{x}{2} f'\left(\frac{x^3}{3}\right) > f'(3 - x^2)$

26. $f'(x) \geq 0$, $g'(x) \leq 0$

$\therefore \quad h'(x) = f'(\underset{>0}{g(x)}\; \underset{<0}{g'(x)}) \leq 0$

$\therefore \quad h(2) = 1$ as $h(1) = 1$

27. $y = a \log|x| + bx^2 + x$

$\therefore \quad \frac{dy}{dx} = \frac{a}{x} + 2bx + 1 = 0$

$\therefore \quad \frac{a + 2bx^2 + x}{x} = 0 \qquad \alpha = \frac{-4}{3}, \beta = 2$

$\therefore \quad \alpha + \beta = \frac{2}{3} = \frac{-1}{2b}$

$\therefore \quad b = \frac{-3}{4}$

$\alpha\beta = \frac{-8}{3} = \frac{a}{2b}$

$\therefore \quad a = \frac{-8}{3} \times 2 \times \frac{-3}{4} = 4$

28. Point on $y^2 = 4x$ is $(t^2, 2t)$

Distance between point & (2,1) is

$d = \sqrt{(t^2-2)^2 + (2t-1)^2}$

$d^2 = (t^2-2)^2 + (2t-1)^2$

$= t^4 - 4t + 5 \qquad = f(t)$

$\therefore \quad f'(t) = 4t^3 - 4 = 0$

$\therefore \quad t = 1$ we can show that $t = 1$ is minima

$\therefore \quad$ Point is $(1,2)$.

28. Point nearest to the required line will have common normal.

$\therefore \quad \frac{dy}{dx} = 3 = 2x + 7$

$\therefore \quad x = -2, y = -8$

point is $(-2,-8)$

30. $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$

$x = a\cos\theta, y = 2\sin\theta$

$\sqrt{a^2\cos^2\theta + 4(1-\sin\theta)^2} = d$

$d^2 = f(\theta) = a^2\cos^2\theta + 4 - 8\sin\theta + 4\sin^2\theta$

$= a^2 + 4 + (4 - a^2)\sin^2\theta - 8\sin\theta$

$\therefore \quad f'(\theta) = 2(4 - a^2)\sin\theta\cos\theta - 8\cos\theta = 0$

$\therefore \quad \cos\theta = 0$

$\Rightarrow \quad \theta = \frac{\pi}{2}$

$\therefore \quad$ point is (0,2).

31. $r\theta + 2r = k$

$\therefore \quad r(\theta + 2) = k$

$\therefore \quad \theta = \frac{k - 2r}{r}$

$\therefore \quad A = \frac{1}{2} r^2 \times \frac{(k-2r)}{r}$

$= \frac{kr - r^2}{2}$

$\therefore \quad \frac{dA}{dr} = 0$

$\therefore \quad r = \frac{k}{4}$

$\therefore \quad \theta = \frac{k - \frac{k}{2}}{\frac{k}{4}} = 2^c$

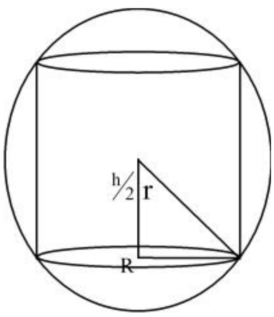
32. From above example, $\theta = 2^c$

$\therefore \quad 2r + 2r = 20$

$\therefore \quad r = 5$

$\therefore \quad A = \frac{1}{2} \times 25 \times 2 = 25$ sq.cm.

33.



$R^2 + \frac{h^2}{4} = r^2$

$v = \pi R^2 h$

$= \pi h \left(r^2 - \frac{h^2}{4} \right)$

$= \pi r^2 h - \frac{\pi h^3}{4}$

$\therefore \quad \frac{dV}{dh} = \pi r^2 - \frac{3\pi h^2}{4} = 0$

$\therefore \quad h^2 = \frac{4r^2}{3} \qquad h = \frac{2r}{\sqrt{3}}$

34. $s = 2\pi r(r+h)$

$= 2\pi r\left(r + \frac{v}{\pi r^2}\right)$

$= 2\pi r^2 + \frac{2v}{r}$

$\frac{ds}{dr} = 0 \qquad \therefore \qquad 4\pi r - \frac{2v}{r^2} = 0$

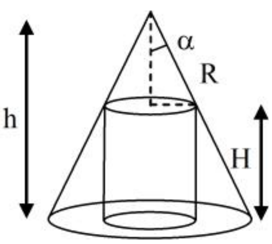
$\therefore \qquad r^3 = \frac{v}{2\pi} \qquad \therefore \qquad r = \left(\frac{v}{2\pi}\right)^{\frac{1}{3}}$

$\pi r^2 = \left(\frac{\pi v^2}{4}\right)^{\frac{1}{3}}$

$\frac{v}{\pi r^2} = \left(\frac{4v}{\pi}\right)^{\frac{1}{3}} = h$

$h = 2r$

35. $\frac{R}{h-H} = \tan\alpha$



$R = \tan\alpha(h-H)$

curved surface area

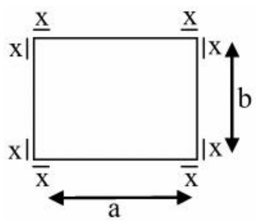
$= S_c = 2\pi RH$

$= 2\pi\tan\alpha\left(hH - H^2\right)$

$\therefore \qquad \frac{dS_c}{dH} = 2\pi\tan\alpha(h-2H) = 0$

$\therefore \qquad H = \frac{h}{2}\cdot$

36.



$V = (a-2x)(b-2x)x$

$V = 4x^3 - 2(a+b)x^2 + abx$

$\therefore \qquad \frac{dV}{dx} = 12x^2 - 4(a+b)x + ab = 0$

$\therefore \quad x = \dfrac{4(a+b) \pm \sqrt{16(a+b)^2 - 48ab}}{24}$

$= \dfrac{(a+b) \pm \sqrt{a^2 + b^2 - ab}}{6}$

But $\quad x < a , x < b$

$\therefore \quad x = \dfrac{(a+b) - (a^2 + b^2 - ab)^{\frac{1}{2}}}{6} = \dfrac{1}{6}\left[(a+b) - (a^2 + b^2 - ab)^{\frac{1}{2}}\right]$

37. $v = x(a - 2x)^2$

$\therefore \quad v = 4x^3 - 4ax^2 + a^2x$

$\therefore \quad \dfrac{dv}{dx} = 12x^2 - 8ax + a^2 = 0 \qquad \therefore \quad x = \dfrac{a}{2} \quad \text{or} \quad x = \dfrac{a}{6}$

But $x = \dfrac{a}{2}$ will make volume zero.

$\therefore \quad x = \dfrac{a}{6}$

38. $a^2h = 32$

$a^2 + 4ah$ has to be minimised

$\therefore \quad h = \dfrac{32}{a^2}$

$\therefore \quad f(a) = a^2 + \dfrac{128}{a}$

$\therefore \quad f'(a) = 2a - \dfrac{128}{a^2} = 0$

$\therefore \quad a = 4 \;\; \& \;\; h = 2$

$\therefore \quad \text{Area} = 16 + 32 = 48$

39. Line is $(y-4) = m(x-3)$

$x = 0 \quad \Rightarrow \quad y = 4 - 3m$

$y = 0 \quad \Rightarrow \quad x = 3 - \dfrac{4}{m}$

$\Delta = \dfrac{1}{2}(4 - 3m)\left(3 - \dfrac{4}{m}\right)$

$= \dfrac{1}{2}\left(24 - 9m + \dfrac{16}{m}\right)$

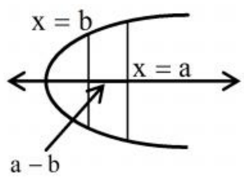
$\therefore \quad \dfrac{d\Delta}{dm} = \dfrac{-9}{2} + \dfrac{16}{2m^2} = 0$

$\therefore \quad \frac{8}{m^2} = \frac{9}{2}$

$\therefore \quad m^2 = \frac{16}{9}$

$\therefore \quad m = \frac{-4}{3}$ as $m > 0$ $\Rightarrow$ no Δ is formed

$\therefore \quad \Delta = \frac{1}{2}(8)(6) = 24$

40. 

$x = b \quad \Rightarrow \quad y^2 = 4ab \qquad y = \pm\sqrt{4ab} = \pm 2\sqrt{ab}$

$\therefore \quad |2y| = \pm 4\sqrt{ab}$

$A = \frac{1}{2}(a-b)\left(4a + 4^2\sqrt{ab}\right)$

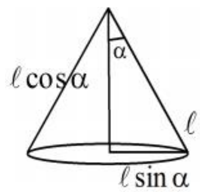
$= 2a^2 + 2a^{\frac{3}{2}}b^{\frac{1}{2}} - 2ab - 2a^{\frac{1}{2}}b^{\frac{3}{2}}$

$\frac{dA}{sb} = \frac{a^{\frac{3}{2}}}{b^{\frac{1}{2}}} - 2a - 3a^{\frac{1}{2}}b^{\frac{1}{2}} = 0$

$\therefore \quad -3a^{\frac{1}{2}}b - 2ab^{\frac{1}{2}} + a^{\frac{3}{2}} = 0$

$\Rightarrow \quad b = \frac{a}{9}$

41. $\therefore \quad V = \frac{1}{3}\pi\ell^3 \sin^2\alpha\cos\alpha$



$\frac{dV}{d\alpha} = \frac{1}{3}\pi\ell^2\left[2\sin\alpha\cos^2\alpha - \sin^3\alpha\right] = 0$

$\therefore \quad \sin\alpha = 0$ or $2\cos^2\alpha = \sin^2\alpha$
Rejected

$\therefore \quad \tan\alpha = \sqrt{2}$ as $\alpha \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad \alpha = \tan^{-1}\left(\sqrt{2}\right)$

42. $V = \frac{1}{3}\pi r^2 h = \text{constant}$

$S_c = \pi r\sqrt{r^2 + h^2}$ has to be maximized

$\therefore \quad h = \frac{3V}{\pi r^2}$

$\therefore \quad S_c = \pi r^2 \sqrt{r^2 + \frac{9V^2}{\pi^2 r^4}}$

$= \sqrt{\pi^2 r^4 + \frac{9V^2}{r^2}}$

$\frac{dS_c}{dr} = \frac{dS_c^2}{dr} = 0$

$\therefore \quad 4\pi^2 r^3 - \frac{18V^2}{r^3} = 0$

$\therefore \quad 4\pi^2 r^6 = 18V^2$

$\therefore \quad r^6 = \frac{9V^2}{2\pi^2}$

$\therefore \quad r = \frac{3^{\frac{1}{3}}}{2^{\frac{1}{6}}} \frac{V^{\frac{1}{3}}}{\pi^{\frac{1}{3}}}$

$r^2 = \left(\frac{9V^2}{2\pi^2}\right)^{\frac{1}{3}}$

$h = \frac{3V}{\pi r^2} = \frac{3V}{\pi} \times \left(\frac{2\pi^2}{9V^2}\right)^{\frac{1}{3}}$

$= \frac{3V}{\pi} \times \frac{2^{\frac{1}{3}}\pi^{\frac{2}{3}}}{9^{\frac{1}{3}}V^{\frac{2}{3}}} = \frac{\left(3^{1/3}\right)\left(2^{1/3}\right)V^{1/3}}{\pi^{1/3}}$

$\frac{h}{r} = 2^{1/3} \times 2^{1/6} = \sqrt{2}$

43. $\ell^2 = h^2 + r^2$

$r = \sqrt{\ell^2 - h^2}$

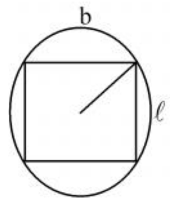
$V = \frac{1}{3} r^2 h = \frac{1}{3}\left(\ell^2 - h^2\right)h$

$= \frac{\ell^2 h}{3} - \frac{h^3}{3}$

$\therefore \quad \frac{dV}{dh} = \frac{\ell^2}{3} - h^2 = 0$

$\therefore \quad h = \frac{\ell}{\sqrt{3}}$

44.



$b^2 + \ell^2 = 4r^2$

$b = \sqrt{4r^2 - \ell^2}$

$S = kb\ell^3$

$\quad = k\ell^3 \sqrt{4r^2 - \ell^2}$

$\therefore \quad \frac{dS}{d\ell} = 3k\ell^2 \sqrt{4r^2 - \ell^2} - \frac{k\ell^4}{\sqrt{4r^2 - \ell^2}} = 0$

$\therefore \quad 3k\ell^2\left(4r^2 - \ell^2\right) = k\ell^4$

$\therefore \quad \ell = 0$ Rejected or

$3\left(4r^2 - \ell^2\right) = \ell^2$

$\therefore \quad 12r^2 = 4\ell^2$

$\therefore \quad \ell = \sqrt{3r}$

$\therefore \quad b = r$

45. $b^2 + d^2 = 4r^2$

$d^2 = 4r^2 - b^2$

$\therefore \quad S = k\,b\,d^2$

$\quad = kb\left(4r^2 - b^2\right) = 4kbr^2 - kb^3$

$\therefore \quad \frac{dS}{dr} = 4kr^2 - 3kb^2 = 0$

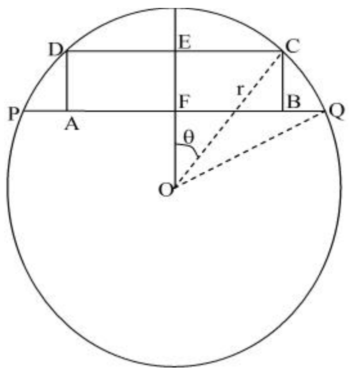
$\therefore \quad b = \frac{2r}{\sqrt{3}}$

$\therefore \quad d^2 = 4r^2 - \frac{4r^2}{3} = \frac{8r^2}{3}$

$\Rightarrow \quad d = \frac{2\sqrt{2}\,r}{3}$

$\therefore \quad d = \sqrt{2}\,b = 2\sqrt{\frac{2}{3}}r$

46. Let OABC be the sheet of paper as



The corner A of the rectangular sheet OABC is folded over along PQ so as to reach the opposite edge OC at R.
Let the crease PQ be of length x.
Let $\angle APQ = \theta$. Then $\angle PQR = \theta$ and $\angle OPR = \pi - 2\theta$.
In ΔAPQ , we have

$$\cos\theta = \frac{AP}{PQ}$$

$\Rightarrow \quad AP = x\cos\theta$
In ΔOPR, we have

$$\cos(\pi - 2\theta) = \frac{OP}{RP}$$

$\Rightarrow \quad -\cos 2\theta = \frac{OP}{AP} \qquad [\because \ AP = RP]$

$\Rightarrow \quad OP = -AP\cos 2\theta = -x\cos\theta\cos 2\theta$
Now,

$a = OA = OP + AP$

$\Rightarrow \quad a = x\cos\theta - x\cos 2\theta\cos\theta$

$\Rightarrow \quad x = \dfrac{a}{\cos\theta - \cos\theta\cos 2\theta} \qquad$ ...(i)

$\Rightarrow \quad \dfrac{a}{x} = \cos\theta - \cos\theta\cos 2\theta$

Let $y = \frac{a}{x}\cdot$ Then y is maximum when x is minimum.
Now,
$y = \cos\theta - \cos\theta\cos 2\theta$

$\Rightarrow \quad \dfrac{dy}{d\theta} = -\sin\theta + \sin\theta\cos 2\theta + 2\cos\theta\sin 2\theta$

For maximum or minimum values of y we must have $\frac{dy}{d\theta} = 0$

$\Rightarrow \quad -\sin\theta + \sin\theta\cos 2\theta + 4\sin\theta\cos^2\theta =$

$\Rightarrow \quad -\sin\theta(1 - \cos 2\theta) + 4\sin\theta(1 - \sin^2\theta) = 0$

$\Rightarrow \quad -2\sin^3\theta + 4\sin\theta - 4\sin^3\theta = 0$

$\Rightarrow \quad 4\sin\theta = 6\sin^3\theta$

$\Rightarrow \quad \sin^2\theta = \dfrac{2}{3}$ or $\sin\theta = 0$

$\Rightarrow \quad \sin\theta = \sqrt{\dfrac{2}{3}}$ or $\theta = 0.$

Now,

$$\frac{d^2y}{d\theta^2} = -\cos\theta + \cos\theta\cos 2\theta - 2\sin\theta\sin 2\theta + 4\cos\theta\cos 2\theta - 2\sin\theta\sin 2\theta$$

$$\Rightarrow \quad \frac{d^2y}{d\theta^2} = -\cos\theta + 5\cos\theta\cos 2\theta - 4\sin\theta\sin 2\theta$$

For $\sin\theta = \sqrt{\dfrac{2}{3}}$ and $\cos\theta = \sqrt{\dfrac{2}{3}}$, we have

$$\frac{d^2y}{d\theta^2} = -\frac{1}{\sqrt{3}} + 5\times\sqrt{\frac{2}{3}}\times\left(\frac{2}{3}-1\right) - 4\times\sqrt{\frac{2}{3}}\times 2\sqrt{\frac{2}{3}}\times\sqrt{\frac{1}{3}} < 0.$$

So, y is maximum when $\sin\theta = \sqrt{\dfrac{2}{3}}$

Hence, x is minimum when $\sin\theta = \sqrt{\dfrac{2}{3}}$

Putting $\sin\theta = \sqrt{\dfrac{2}{3}}$ and $\cos\theta = \sqrt{\dfrac{1}{3}}$ in (i), we get

$$\text{Length of the crease} = x = \frac{a}{\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}\left(1 - 2\times\frac{2}{3}\right)} = \frac{3\sqrt{3}a}{4}$$

47. Let speed of boat be v & walking speed be v sec α

$$\therefore \quad t = \frac{\sqrt{a^2 + (b-x)^2}}{v} + \frac{x\cos\alpha}{v}$$

$$= \frac{\sqrt{a^2 + (b-x)^2} + x\cos\alpha}{v}$$

$$\therefore \quad v\frac{dt}{dx} = \cos\alpha + \frac{1}{2\sqrt{a^2 + (b-x)^2}} \times -2(b-x) = 0$$

$$\therefore \quad \cos\alpha = \frac{b-x}{\sqrt{a^2 + (b-x)^2}}$$

$$\therefore \quad (b-x)^2 = \cos^2\alpha\left(a^2 + b^2 - 2bx + x^2\right)$$

$\therefore \quad (b-x)^2 = a^2 \cot^2 \alpha$

$\therefore \quad x = b - a\cot\alpha = \dfrac{b\sin\alpha - a\cos\alpha}{\sin\alpha}$

48. $\therefore \quad T = \dfrac{\sqrt{d^2+x^2}}{u} + \dfrac{1-x}{v}$

$\therefore \quad \dfrac{dT}{dx} = \dfrac{x}{u\sqrt{d^2+x^2}} - \dfrac{1}{v} = 0$

$\therefore \quad xv = u\sqrt{d^2+x^2}$

$\therefore \quad x^2\left(v^2-u^2\right) = u^2d^2$

$\therefore \quad x = \dfrac{ud}{\sqrt{v^2-u^2}}$

For students. [Think for solution if $u > v$]

49. $2\ell + 2\pi r = 440$

$\therefore \quad \ell + \pi r = 220 \quad \& \quad \ell = 220 - \pi r$

$A = 2\left(220r - \pi r^2\right) \qquad = 2\ell r$

$\dfrac{dA}{dr} = 2\left(220 - 2\pi r\right) = 0$

$\therefore \quad r = 35 \text{ ft}$

$\Rightarrow \quad 2r = 70\text{ft} \quad \& \quad \ell = 110\text{ft}$

50. $P(x) = a_0 + a_1x^2 + a_2x^4 + \ldots\ldots + a_nx^{2n}$

$0 < a_1 < a_2 < \ldots.. < a_n$

$\therefore \quad P'(x) = 2na_nx^{2n-1} + \ldots. + 4a_2x^3 + 2a_1x$

$= 0$ only at $x = 0$ &

$P''(x) > 0 \quad \forall \; x \in R$

$\therefore \quad P(x)$ has only one minimum.

51. $x = a\sec\theta, y = b\,\text{cosec}\,\theta$

Minimum radius vector = ?

$r = \sqrt{x^2+y^2} = \sqrt{a^2\sec^2\theta + b^2\,\text{cosec}^2\,\theta}$

From (Q.4),

Minimum value of $r = \sqrt{(a+b)^2} = a+b$

52. From (Q.18)

$s = 2\pi r(r+h)$

$$= 2\pi r\left(r + \frac{v}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2v}{r}$$

$$\frac{ds}{dr} = 0 \qquad \therefore \qquad 4\pi r - \frac{2v}{r^2} = 0$$

$$\therefore \qquad r^3 = \frac{v}{2\pi} \qquad \therefore \qquad r = \left(\frac{v}{2\pi}\right)^{\frac{1}{3}}$$

$$\pi r^2 = \left(\frac{\pi v^2}{4/3}\right)^{\frac{1}{3}}$$

$$\frac{v}{\pi r^2} = \left(\frac{4v}{\pi}\right) = h$$

$h = 2r$, form this statement $h : r = 2 : 1$

53. $f(x) = (x-1)^p (x-2)^q$

$$\therefore \qquad f'(x) = p(x-1)^{p-1}(x-2)^q + q(x-1)^p(x-2)^{q-1}$$

$$f''(x) = p(p-1)(x-1)^{p-2}(x-2)^q + 2pq(x-1)^{p-1}(x-2)^{q-1} + q(q-1)(x-1)^p(x-2)^{q-2}$$

If we go on taking derivatives, we find that the condition given in the question holds when (even)th derivative is non-zero for it, p & q should be even.

54. $f(x) = xe^x$

$$f'(x) = xe^x + e^x = 0$$

$$\therefore \qquad x = -1$$

$f''(x) = xe^x + 2e^x > 0$ for $x = -1$

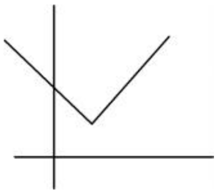
$\therefore \qquad x = -1$ is a minimum

55. Time required $= T = \left(\frac{N}{x}\right)(\alpha + \beta x^2)$

$$\therefore \qquad T = N\left(\frac{\alpha}{x} + \beta x\right)$$

$$\therefore \qquad \frac{dT}{dx} = N\left(\beta - \frac{\alpha}{x^2}\right) = 0 \qquad \therefore \qquad x = \sqrt{\frac{\alpha}{\beta}}$$

56. $f(x) = \max\{x, x+1, 2-x\}$

By graph, $f(x) =$ 

$$f(x) = 2 - x, x \le +\frac{1}{2}$$

$$x + 1, x > \frac{1}{2}$$

$\therefore$ $x = \frac{1}{2}$ is point of minima and minimum value is $\frac{3}{2}$.

57. $f(\alpha) = \left(1 + \frac{1}{\sin^n x}\right)\left(1 + \frac{1}{\cos^n \alpha}\right)$

$$= (1 + \sec^n \alpha)(1 + \operatorname{cosec}^n \alpha)$$

$$= 1 + \sec^n \alpha + \operatorname{cosec}^n \alpha + \sec^n \alpha \operatorname{cosec}^n \alpha$$

$\therefore$ $f'(\alpha) = n \sec^n \alpha \tan \alpha - n \operatorname{cosec}^n \alpha \cot \alpha$

$$+n \sec^n \alpha \operatorname{cosec}^n \alpha (\tan \alpha - \cot \alpha) = 0$$

$\therefore$ $\sec^n \alpha \tan \alpha (1 + \operatorname{cosec}^n \alpha) = \operatorname{cosec}^n \alpha \cot \alpha (1 + \sec^n \alpha)$

$\therefore$ $\frac{(\sec^n \alpha)(\sec^2 \alpha - 1)}{1 + \sec^n \alpha} = \frac{\operatorname{cosec}^n \alpha}{1 + \operatorname{cosec}^n \alpha}$

$\therefore$ $\frac{(\cos^n \alpha)(\sin^2 \alpha)}{(\cos^n \alpha + 1)(\cos^2 \alpha)} = \frac{\sin^n \alpha}{1 + \sin^n \alpha}$

$\therefore$ $\frac{\sin^{n-2} \alpha}{1 + \sin^n \alpha} = \frac{\cos^{n-2} \alpha}{1 + \cos^n \alpha}$

$\Rightarrow$ $\sin \alpha = \cos \alpha$

For minima,

$$\sin \alpha = \cos \alpha = \frac{+1}{\sqrt{2}}$$

$\therefore$ Minimum value $= (1 + 2^{n/2})^2$

58. $f(x) + f\left(\frac{1}{x}\right) = \frac{1}{x}$

$\therefore$ $f(x) + f\left(\frac{1}{x}\right) = x$

$\Rightarrow$ $x = \frac{1}{x}$

$\Rightarrow \quad x = \pm 1$ Only

Here, $x = -1 \quad \Rightarrow \quad f(x) = -\frac{1}{2}$ and

$x = +1 \quad \Rightarrow \quad f(x) = +\frac{1}{2}$

$\therefore \quad f(x)$ has maximum value $\frac{1}{2}$.

59. $f(x) = \cos 2\pi x + \{x\}$

At non-integral points,

$f'(x) = -2\pi \sin 2\pi x + 1$

It tends to achieving maximum values at points infinitesimally close to and less than integers but it has a discontinuity.

$\therefore$ It has no maxima.

60. $f(x) = x - x^2$

$x_1 \,\&\, x_2 \in \quad y = x - x^2 \quad$ in $(0,1)$

maximum value of expression

$= \max\left(x - x^2\right) = \frac{1}{4}$

61. $f(x) = x^2, \qquad x \in [-2,-1] \cup [1,2]$

$2 - x^2, \qquad x \in (-1,1)$

$\therefore$ Function has maximum at $x = 0$ & local as well as global minima at $x = \pm 1$

62. $x^3 - ax^2 + bx - 6 = 0$ has roots real and positive

$\therefore \quad \alpha\beta\gamma = 6, \alpha + \beta + \gamma = a, \alpha\beta + \beta\gamma + \gamma\alpha = b$

$\alpha\beta + \beta\gamma + \gamma\alpha = b$

$\therefore \quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{b}{6}$

Now, sum is minimum when each of them is equal

$$\frac{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}}{3} \geq \left(\frac{1}{\alpha\beta\gamma}\right)^{\frac{1}{3}} \qquad \text{[AM-GM inequality]}$$

$\therefore \quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \geq \frac{3}{6^{1/3}} \qquad \therefore \quad b \geq \frac{3 \times 6}{6^{1/3}} = 3(36)^{1/3}$

63. $f'(x) = \dfrac{2}{3(6-x)^{\frac{1}{3}}}$

Which is not diff. at $x = 6$

$\therefore$ Theorems are not applicable.

64. By definition.

65. $f(0) = -6$, $f(4) = +6$

$\therefore \quad f'(x) = (x-2)(x-3) + (x-1)(x-2) + (x-1)(x-3)$

$f'(c) = \frac{6+6}{4-0} = 3$

$\therefore \quad 3x^2 - 12x + 11 = 3$ and $x = c$

$\therefore \quad 3c^2 - 12c + 8 = 0$

$c = 12 \pm \frac{\sqrt{144-96}}{6}$

$= 12 \pm \frac{\sqrt{48}}{6} = 6 \pm \frac{2\sqrt{3}}{3} = 2 \pm \frac{2}{\sqrt{3}}$

66. $f(x) = x^{\alpha} \log x$

$f'(x) = x^{\alpha-1}(1 + \alpha \log x) = 0$

$c = e^{-1/\alpha} \in (0,1)$

$\therefore \quad \alpha > 0$

67. $a + b + c = 0$

$f(x) = ax^3 + bx^2 + cx$

Has roots 0 & 1

$\therefore \quad 3ax^2 + 2bx + c = 0$

has at least one root in $(0,1)$.

68. $f'(c) = \frac{13-5}{2} = 4$

69. Refer (Q.28) (above)

$a + b + c = 0$

$f(x) = ax^3 + bx^2 + cx$

Has roots 0 & 1

$\therefore \quad 3ax^2 + 2bx + c = 0$

70. $x^3 - 3x + a = 0$ has two roots in $[0,1]$

$f'(x) = 3x^2 - 3 \neq 0$ in $(0,1)$

$\therefore$ There is no value of a satisfying the conditions.