Applications Of Derivatives
EXERCISE 1(B)

1. $y = 2x^3 + 13x^2 + 5x + 9$

$\therefore \quad \dfrac{dy}{dx} = 6x^2 + 26x + 5$

Let the point be (h, k)

$\therefore \quad k = 2h^3 + 13h^2 + 5h + 9$

$\therefore \quad \dfrac{y - \left(2h^3 + 13h^2 + 5h + 9\right)}{x - h} = 6h^2 + 26h + 5$

substituting $(0, 0)$

$\therefore \quad 2h^3 + 13h^2 + 5h + 9 = 6h^3 + 26h^2 + 5h$

$\therefore \quad 4h^3 + 13h^2 - 9 = 0$

$\Rightarrow \quad h = -1, k = 15.$

2. $x = a\left(t + \sin t \cos t\right)$

$y = a\left(1 + \sin t\right)^2$

$\therefore \quad \dfrac{dx}{dt} = a\left(1 + \cos 2t\right)$

$\dfrac{dy}{dt} = 2a\left(1 + \sin t\right)\cos t$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{2\cos t + \sin 2t}{1 + \cos 2t}$

$$= \frac{2\cos t\left(1 + \sin t\right)}{2\cos^2 t} = \frac{\left(\cos\frac{t}{2} + \sin\frac{t}{2}\right)^2}{\cos^2\frac{t}{2} - \sin^2\frac{t}{2}}$$

$$= \frac{1 + \tan\frac{t}{2}}{1 - \tan\frac{t}{2}} \qquad = \tan\left(\frac{\pi}{4} + \frac{t}{2}\right)$$

3. $\left(\dfrac{x}{a}\right)^n + \left(\dfrac{y}{b}\right)^n = 2$

$\therefore \quad \dfrac{nx^{n-1}}{a^n} + \dfrac{ny^{n-1}}{b^n}\dfrac{dy}{dx} = 0$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{b^n x^{n-1}}{a^n y^{n-1}}$

At (a, b)

$$\frac{dy}{dx} = -\frac{b}{a}$$

$\therefore \quad b(x-a) + a(y-b) = 0$

$\therefore \quad bx + ay = 2ab$

$\therefore \quad \frac{x}{a} + \frac{y}{b} = 2.$

4. $y = be^{-x/a}$

$x = 0 \quad \Rightarrow \quad y = b$

$y' = -\frac{b}{a}e^{-x/a}$

$\therefore \quad \left.\frac{dy}{dx}\right|_{(0,b)} = -\frac{b}{a}$

$\therefore \quad a(y-b) = -bx$

$\therefore \quad bx + y = ab$

$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1.$

5. $y = 3x^2 + bx + 2$

$x = 0 \quad \Rightarrow \quad y = 2$

$\frac{dy}{dx} = 6x + b$

$\therefore \quad \left.\frac{dy}{dx}\right|_{x=0} = b = 4$ .... Given

$\therefore \quad b = 4$

6. $y = \frac{8 - x^2}{2}$

$\therefore \quad \frac{dy}{dx} = -x = -2$ [Given]

$\therefore \quad x = 2$

$\Rightarrow \quad y = 2$

$\therefore \quad y - 2 + 2(x-2) = 0$

$\therefore \quad 2x + y - 6 = 0.$

7. Points are $(p, ap^2 + bp + c)$ & $(q, aq^2 + bq + c)$

$\therefore \quad \frac{y - (ap^2 + bp + c)}{x - p} = \frac{a(q^2 - p^2) + b(q - p)}{q - p}$

$$\therefore \quad \frac{y-\left(ap^2+bp+c\right)}{x-p}=aq+ap+b$$

$$\therefore \quad y=\left(aq+ap+b\right)x-apq+c$$

$$\therefore \quad m=aq+ap+b$$

$$\& \quad m=2ax+b=\frac{dy}{dx}$$

$$\therefore \quad x=\frac{p+q}{2}$$

8. $\sqrt{x}+\sqrt{y}=\sqrt{a}$

$$\therefore \quad \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0 \qquad x-x=\sqrt{xy}$$

$$\therefore \quad \frac{dy}{dx}=-\sqrt{\frac{y}{x}}$$

$$\therefore \quad \frac{Y-y}{X-x}=-\sqrt{\frac{y}{x}}$$

$$X=0 \quad \Rightarrow \quad Y=y+\sqrt{xy}$$

$$Y=0 \quad \Rightarrow \quad X=x+\sqrt{xy}$$

$$x+y+2\sqrt{xy}=OA+OB=\left(\sqrt{x}+\sqrt{y}\right)^2=a$$

9. $x^{2/3}+y^{2/3}=a^{2/3}$

$$\therefore \quad \frac{2}{3x^{1/3}}+\frac{2}{3y^{1/3}}\frac{dy}{dx}=0$$

$$\therefore \quad \frac{dy}{dx}=-\left(\frac{y}{x}\right)^{1/3}$$

$$\therefore \quad \frac{Y-y}{X-x}=-\left(\frac{y}{x}\right)^{1/3}$$

$$\therefore \quad Y=y+x^{2/3}y^{1/3} \qquad \text{When } X=0$$

$$X=x+x^{1/3}+y^{2/3} \qquad \text{When } Y=0$$

$$Y^2+X^2=y^2+x^{4/3}y^{2/3}+2x^{2/3}y^{4/3}+x^2+2x^{4/3}y^{2/3}+x^{2/3}y^{4/3}$$

$$=x^2+y^2+3x^{4/3}y^{2/3}+3x^{2/3}y^{4/3}$$

$$=x^2+x^{4/3}y^{2/3}+y^2+x^{2/3}y^{4/3}+2\left(x^{4/3}y^{2/3}+x^{2/3}y^{4/3}\right)$$

$$=\left(x^{4/3}+y^{4/3}\right)a^{2/3}+\left(2x^{2/3}y^{2/3}\right)a^{2/3}$$

$$=a^{2/3}\left(x^{2/3}+y^{2/3}\right)$$

$$=a^2$$

10. $xy^n = a^{n+1}$

$\therefore \quad y^n = nxy^{n-1}\dfrac{dy}{dx} = 0 \quad , \; n \neq -1$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{y}{nx}$

$\therefore \quad \dfrac{Y-y}{X-x} = -\dfrac{y}{nx}$

$X = 0 \;\Rightarrow\; Y = y + \dfrac{y}{n}$

$Y = 0 \;\Rightarrow\; X = x + nx$

$\therefore \quad \Delta = \dfrac{1}{2}XY = \dfrac{1}{2}xy(1+n)\left(1+\dfrac{1}{n}\right)$, Here n is a constant.

Δ is constant only when xy is constant but xy^n is constant

$\therefore \quad n = 1.$

11. $f(x) = 2x^3 - 9x^2 + 12x - 3$

$\therefore \quad f'(x) = 6x^2 - 18x + 12 > 0$

$\therefore \quad x^2 - 3x + 2 > 0$

$\therefore \quad x \in (-\infty, 1) \cup (2, \infty)$

12. $f(x) = x^3 - ax^2 + 48x + 19$

$f'(x) = 3x^2 - 2ax + 48 \geq 0 \quad \forall\, x$

$\therefore \quad (2a)^2 - 4(3)(48) \leq 0$

$\therefore \quad a^2 - 144 \leq 0$

$\therefore \quad a \in [-12, 12]$

13. $f(x) = 2x^3 - 9x^2 - 60x + 81$

$\therefore \quad f'(x) = 6x^2 - 18x - 60 < 0$

$\therefore \quad x^2 - 3x - 10 < 0$

$\therefore \quad x \in (-2, 5)$

14. $f(x) = \dfrac{x^2}{x+2}$

$\therefore \quad f'(x) = \dfrac{2x(x+2) - x^2}{x+2}$

$= \dfrac{x^2 + 4x}{x+2} < 0$

$\therefore \quad \frac{x(x+4)}{(x+2)} < 0 \qquad \therefore \quad x \in (-\infty,-4) \cup (-2,0)$

15. $f(x) = x^2$

$\therefore \quad f'(x) = x^x (1 + \ell n\, x) = 0$

$\therefore \quad 1 + \ell n\ x = 0$

$\therefore \quad x = \frac{1}{e}$

For $x < \frac{1}{e}, f'(x) < 0$

$\therefore$ Function decreases in $\left(0, \frac{1}{e}\right)$.

16. $f(x) = \frac{\log x}{x}$

$\therefore \quad f'(x) = \frac{1 - \log x}{x^2} < 0$

$\therefore \quad \log x > 1 \quad \Rightarrow \quad x > e$

$\therefore \quad x \in (e, \infty)$

17. $f(x) = 2|x-2| + |x-3|$

For $x < 2$

$f(x) = 2(2-x) + 3 - x$

$= 7 - 3x$ is a decreasing function.

For $2 < x < 3$

$f(x) = 2(x-2) + 3 - x$

$= x - 1$ is increasing function.

For $x > 3$

$f(x) = 3x - 7$ is an increasing function.

$\therefore \quad x \in (2, \infty)$

18. $f(x) = \cos x - \sin x$

$\therefore \quad f'(x) = -\sin x - \cos x < 0 \qquad \forall\ x \in \left(0, \frac{\pi}{2}\right)$

$g(x) = \cos x + \sin x$

$g'(x) = \cos x - \sin x > 0 \qquad \forall\ x \in \left(0, \frac{\pi}{4}\right)$

$< 0 \qquad \forall\ x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$$h(x) = \frac{\sin x}{x}$$

$$h'(x) = \frac{x\cos x - \sin x}{x^2} = 0 \quad \text{at} \quad x = 0,$$

$$h''(x) = \frac{x^2(-x\sin x) - 2x(x\cos x - \sin x)}{x^2} < 0 \qquad \forall\, x \in \left(0, \frac{\pi}{2}\right)$$

$\therefore \quad h'(x) < 0 \qquad \forall\, x \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad \frac{x}{\sin x}$ being reciprocal of $\frac{\sin x}{x}$ is an increasing function.

19. $f(x) = \dfrac{a\sin x + b\cos x}{c\sin x + d\cos x}$

$$f'(x) = \frac{(a\cos x - b\sin x)(c\sin x + d\cos x) - (a\sin x + b\cos x)(c\cos x - d\sin x)}{(c\sin x + d\cos x)^2}$$

$$= \frac{ad - bc}{(c\sin x + d\cos x)^2} > 0 \qquad \forall\, x \qquad \text{iff} \qquad ad - bc > 0.$$

20. $\sin x - bx + c = f(x)$

$$f'(x) = \cos x - b \le 0 \quad \forall \; b \ge 1$$

21. $y = 2x^3 - 3x^2 - 36x + 10 = f(x)$

$$f'(x) = 6x^2 - 6x - 36 = 0$$

$\therefore \quad x^2 - x - 6 = 0$

$\therefore \quad x = 3 \;\text{ or }\; x = -2$

$$f(3) = 2(27) - 3(9) - 36(3) + 10$$

$$= -71$$

$$f(-2) = 2(-8) - 3(4) - 36(-2) + 10$$

$$= -28 + 82 = 54$$

22. $f(x) = x^2 - 3x + 3$

$\therefore \quad f'(x) = 2x - 3 = 0$

$\therefore \quad x = \frac{3}{2} \qquad\qquad f''(x) = 2 > 0$

$\therefore \quad f(x)$ has minima at $x = \frac{3}{2}$

$$f\left(\frac{3}{2}\right)=\frac{9}{4}-\frac{9}{2}+3=\frac{3}{4}$$

23. $f(x)=2x^3-3x^2-12x+8$

$f'(x)=6x^2-6x-12=0$

$\therefore \quad x=2 \quad \text{or} \quad x=-1$

$f''(x)=12x-6 \qquad <0 \quad \text{for} \quad x=-1$

and $\qquad >0 \quad \text{for} \quad x=2$

$\therefore \quad x=2$ is minima

24. $a^2\sec^2 x+b^2\operatorname{cosec}^2 x=f(x)$

$f'(x)=2a^2\sec^2 x\tan x-2b^2\operatorname{cosec}^2 x\cot x=0$

$f(x)$ can have minima only as maxima is ∞.

$\therefore \quad a^2\sec^2 x\tan x=b^2\operatorname{cosec}^2 x\cot x$

$\therefore \quad a^2\dfrac{\sin x}{\cos^3 x}=b^2\dfrac{\cos x}{\sin^3 x}$

$\therefore \quad \tan^4 x=\dfrac{b^2}{a^2}$

$\therefore \quad \tan^2 x=\left|\dfrac{b}{a}\right|, \ \cot^2 x=\left|\dfrac{a}{b}\right|$

For $a, b>0$

$\sec^2 x=\dfrac{a+b}{a}, \operatorname{cosec}^2 x=\dfrac{a+b}{b}$

$\therefore \quad a^2\sec^2 x+b^2\operatorname{cosec}^2 x$

$\qquad =a^2+ab+ab+b^2=(a+b)^2$

25. $f(x)=\sin x+\sin x\cos x$

$f'(x)=\cos x+\cos^2 x-\sin^2 x=0 \qquad =\cos x+\cos 2x$

$\therefore \quad \cos x+2\cos^2 x-1=0$

$\therefore \quad \cos x=-1 \quad \Rightarrow \quad x=\pi \quad \text{or} \quad \cos x=\dfrac{1}{2} \quad \Rightarrow \quad x=\dfrac{\pi}{3} \ \text{or} \ -\dfrac{\pi}{3}$

$f''(x)=-\sin x-2\sin 2x \qquad <0 \quad \text{for} \quad x=\dfrac{\pi}{3}$

$\qquad >0 \quad \text{for} \quad x=-\dfrac{\pi}{3}$

$\therefore \quad x=\dfrac{\pi}{3}$

26. $f(x) = x + \sin x$

$f'(x) = 1 + \cos x \qquad f''(x) = -\sin x$

When $f'(x) = 0$, $f''(x) = 0$

$\therefore$ $f(x)$ has neither minimum nor maximum.

27. $$\Delta = \frac{1}{2}\begin{vmatrix} a & 0 & 1 \\ a\cos\theta & b\sin\theta & 1 \\ a\cos\theta & -b\sin\theta & 1 \end{vmatrix}$$

$= \frac{1}{2}(2ab\sin\theta - 2ab\cos\theta\sin\theta)$

$= ab(\sin\theta - \sin\theta\cos\theta)$

$\therefore \quad \frac{d\Delta}{d\theta} = ab(\cos\theta - \cos 2\theta) = 0$

$\therefore \quad 2\cos^2\theta - \cos\theta - 1 = 0$

$\cos\theta = 1 \quad \text{or} \quad \cos\theta = -\frac{1}{2}$

Now for a triangle,as it will be a st.line, $\theta \neq 0$, $\therefore \quad \cos\theta \neq 1$

$\therefore \quad \cos\theta = -\frac{1}{2}$

$\therefore \quad \theta = \frac{2\pi}{3} \qquad \Rightarrow \qquad \sin\theta = \frac{\sqrt{3}}{2}$

$\therefore \quad \Delta = ab\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}\right) = \frac{3\sqrt{3}ab}{4}$

28. $f'(x) = 3x^2 - 6x + 6 \quad > 0 \qquad \text{for all } x$

$\therefore$ Neither minimum nor maximum.

29. $f(x) = (x+6)^4 (8-x)^3$

Let $x + 6 = t$

$\therefore \quad 8 - x = 14 - t$

$\therefore \quad f(t) = t^4 (14-t)^3 \qquad \& \ f'(x) = f'(t)$

$\therefore \quad f'(t) = 4t^3(14-t)^3 - 3t^4(14-t)^2 = 0$

$\therefore \quad t^3(14-t)^2\left[4(14-t) - 3t\right] = 0$

$\therefore \quad t^3(4-t)^2[56 - 7t] = 0$

$\therefore \quad t = 0$ or $t = 14$ or $t = 8$

Now $t \neq 0$ & $t \neq 14$ as product will become zero

$\therefore \quad t = 8$ & $f(t) = 8^4.6^3$

30. $x^2-(a-2)x-(a+1)=0$

$\alpha+\beta=a-2,\ \alpha\beta=-(a+1)$

$\therefore\quad \alpha^2+\beta^2=a^2-4a+4+2a+2$

$\therefore\quad f(a)=a^2-2a+6$

$\therefore\quad f'(a)=2a-2=0 \qquad\qquad f''(a)>0$

$\therefore\quad a=1$

$\therefore\quad f(a)=5=\min(\alpha^2+\beta^2)$

31. Function must be continuous and differentiable to apply Rolle's theorem.

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33. $f(x)=\log(\sin x) \qquad \text{in } \left[\frac{\pi}{6},\frac{5\pi}{6}\right]$

$f\left(\frac{\pi}{6}\right)=f\left(\frac{5\pi}{6}\right)$

$f'(x)=\cot x=0$ at $x=\frac{\pi}{2}$

$\therefore\quad c=\frac{\pi}{2}$

34. $f(x)=\log\left(\frac{x^2+ab}{x(a+b)}\right)$ in $[a,b]$

$f'(x)=\frac{2x}{x^2+ab}-\frac{(a+b)}{x(a+b)}=0$

$\therefore\quad 2x^2=x^2+ab$

$\therefore$ x = GM of a & b.

35. $f(x)=x^3+bx^2+ax$ satisfies Rolle's theorem on $[1,3]$

$c=2+\frac{1}{\sqrt{3}}$

$\therefore\quad f(1)=f(3)$

$\therefore\quad 1+a+b=27+9b+3a$ and $3\left(1+\frac{1}{\sqrt{3}}\right)^2+2b\left(2+\frac{1}{\sqrt{3}}\right)+a=0$

Solving , we get $(a,b)=(11,-6)$

36. $f(x)=\log x$ in $[1,e]$

$f'(x)=\frac{1}{x} \qquad f'(c)=\frac{1}{c}$

$\therefore\quad \frac{1}{c}=\frac{\log e-\log 1}{e-1}=\frac{1}{e-1}$

$\therefore\quad c=e-1.$

37. Here, $f(0) = f(2) = 0$ in $[0,2]$

$\therefore \quad f'(c) = 0$

$\therefore \quad (c-2)^2 + 2c(c-2) = 0$

$\therefore \quad (c-2)[3c-2] = 0$

$\therefore \quad c = \frac{2}{3}$ as $c \in (0,2)$

38. $f(x) = \ell x^2 + mx + n$ in

$\therefore \quad f'(c) = 2\ell x + m = \frac{\ell(b^2 - a^2) + m(b-a)}{b-a}$

$\therefore \quad 2\ell x + m = \ell(b+a) + m$

$\therefore \quad x = \frac{a+b}{2}$

39. Function should be diffrentiable in domain.

40. $\sqrt{x+1} - \sqrt{x} = \frac{1}{\sqrt{x} + \sqrt{x+1}}$

Now, $x > N^2$,

$\therefore \quad x + 1 > N^2$

$\therefore \quad \sqrt{x} > N, \sqrt{x+1} > N$

$\therefore \quad \sqrt{x} + \sqrt{x+1} > 2N$

$\therefore \quad \frac{1}{\sqrt{x} + \sqrt{x+1}} < \frac{1}{2N}$