CONTINUITY & DIFFERENTIABILITY
EXERCISE 3

1 Let $f(x) = \begin{cases} \dfrac{\ln \cos x}{\sqrt[4]{1+x^2}-1} & \text{if } x > 0 \\ \dfrac{e^{\sin 4x}-1}{\ln(1+\tan 2x)} & \text{if } x < 0 \end{cases}$

Is it possible to define f(0) to make the function continuous at x = 0. If yes what is the value of f(0), if not then indicate the nature of discontinuity.

Sol. $\text{LHL}|_{x=0} = \lim\limits_{x\to 0^-} \dfrac{e^{\sin 4x}-1}{\ell n(1+\tan 2x)}$

put x = 0 – h

$= \lim\limits_{x\to 0} \dfrac{e^{-\sin 4x}-1}{\ell n(1-\tan 2h)}$

$= \lim\limits_{h\to 0} \dfrac{e^{-\sin 4h}-1}{-\sin 4h}\left(\dfrac{-\sin 4h}{4h}\right).\,4h\left(\dfrac{1}{\dfrac{\ell n(1-\tan 2h)}{(-\tan 2h)}\left(\dfrac{-\tan 2h}{2h}\right).2h}\right)$

$\boxed{f(0^-) = 2}$

$\text{RHL}|_{x=0} = \lim\limits_{x\to 0^+}\left(\dfrac{\ell n \cos x}{\sqrt[4]{(1+x^2)}-1}\right)$

$= \lim\limits_{x\to 0^+}\left(\dfrac{\cos x - 1}{1+\frac{1}{4}x^2-1}\right)$

$= \lim\limits_{x\to 0^+}\left(\dfrac{1-\cos x}{x^2}\right)(-4)$

$\boxed{f(0^+) = -2}$

hence f(0) can not define.

and ∵ $f(0^-)$ & $f(0^+)$ are finite hence there non-removable type disconti.

2 Let $y_n(x) = x^2 + \dfrac{x^2}{1+x^2} + \dfrac{x^2}{(1+x^2)^2} + \ldots\ldots\ldots\ldots + \dfrac{x^2}{(1+x^2)^{n-1}}$ and $y(x) = \lim\limits_{n\to\infty} y_n(x)$

Discuss the continuity of $y_n(x)$ $(n \in N)$ and y(x) at x = 0

Sol. $y_n(x) = x^2 + \dfrac{x^2}{1+x^2} + \dfrac{x^2}{(1+x^2)^2} + \ldots + \dfrac{x^2}{(1+x^2)^{n-1}}$

$$y_n(x) = x^2 \frac{\left(1-\left(\frac{1}{1+x^2}\right)^n\right)}{1-\frac{1}{1+x^2}}$$

$$= x^2 \frac{\left\{1-\left(\frac{1}{1+x^2}\right)^n\right\}}{\frac{1+x^2-1}{1+x^2}}$$

$y_n(x) = (1 + x^3) \{1 - (1 + x^2)^{-n}\}$

3 Let $f(x) = \begin{cases} \frac{1-\sin \pi x}{1+\cos 2\pi x}, & x < \frac{1}{2} \\ p, & x = \frac{1}{2} \\ \frac{\sqrt{2x-1}}{\sqrt{4+\sqrt{2x-1}}-2}, & x > \frac{1}{2} \end{cases}$. Determine the value of p, if possible, so that the function is continuous at x = 1/2.

Sol. $V.F.|_{x=\frac{1}{2}} = p$ ...(1)

$$LHL|_{x=\frac{1}{2}} = \lim_{x\to\frac{1}{2}^-} f(x)$$

$$= \lim_{x\to\frac{1}{2}^-} \frac{1-\sin \pi x}{1+\cos(2\pi x)}$$

put $x = \frac{1}{2} - h$

$$= \lim_{h\to 0} \frac{1-\sin\left(\frac{\pi}{2}-\pi h\right)}{1+\cos(\pi-2\pi h)}$$

$$= \lim_{h\to 0} \left(\frac{1-\cos \pi h}{(\pi h)^2}\right)\left(\frac{1}{\frac{1-\cos(2\pi h)}{(2\pi h)^2}}\right)\left(\frac{\pi^2 h^2}{4\pi^2 h^2}\right)$$

$$LHL|_{x=\frac{1}{2}} = \frac{1}{4}$$ ...(2)

$\text{RHL}\big|_{x=\frac{1}{2}} = \lim_{x\to\left(\frac{1}{2}\right)^+} f(x)$

$= \lim_{x\to\left(\frac{1}{2}\right)^+} \left(\frac{\sqrt{2x-1}}{\sqrt{4+\sqrt{2x-1}}-2}\right)$

$= \lim_{x\to\frac{1}{2}^+} \left(\frac{\sqrt{2x-1}}{4+\sqrt{2x-1}-4}\right)(\sqrt{4+\sqrt{2x-1}}+2)$

$\text{RHL}\big|_{x=\frac{1}{2}} = 4$

$\because\ \text{LHL}\big|_{x=\frac{1}{2}} \neq \text{RHL}\big|_{x=\frac{1}{2}}$

so the value of function cannot determine & the function is discontinuous.

4 Given the function g (x) = $\sqrt{6-2x}$ and h (x) = $2x^2 - 3x + a$. Then

(a) evaluate h (g(2))   (b) If $f(x) = \begin{cases} g(x), & x \le 1 \\ h(x), & x > 1 \end{cases}$, find 'a' so that f is continuous.

Sol. (i) h(g(2)) =

$g(2) = \sqrt{6-4} = \sqrt{2}$

$h(x) = 2x^2 - 3x + a$

$h(\sqrt{2}) = 4 - 3\sqrt{2} + a$ **Ans**

(ii) $f(x) = \begin{cases} g(x) & ; & x \le 1 \\ h(x) & ; & x > 1 \end{cases}$

$f(x) = \begin{cases} \sqrt{6-2x} & ; & x \le 1 \\ 2x^2 - 3x + a & ; & x > 1 \end{cases}$

$\text{V.F.}\big|_{x=1} = 2$ ...(1)

$\text{R.H.L.}\big|_{x=1} = \lim_{x\to 1^+} f(x)$

$= \lim_{x\to 1^+} (2x^2 - 3x + a)$

$\text{R.H.L.}\big|_{x=1} = a - 1$ ...(2)

$\text{L.H.L.}\big|_{x=1} = \lim_{x\to 1^-} f(x)$

$= \lim_{x\to 1^-} \sqrt{6-2x}$

$= 2$

since function is conti

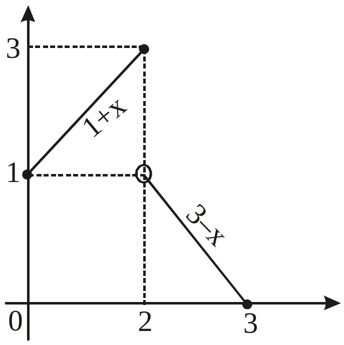
L.H.L.$|_{x=1}$ = R.H.L.$|_{x=1}$ = VF$|_{x=1}$

$2 \quad = a - 1 = 2$

$a - 1 = 2 \Rightarrow \boxed{a = 3}$

5 Let $f(x) = \begin{cases} 1+x & , \ 0 \le x \le 2 \\ 3-x & , \ 2 < x \le 3 \end{cases}$. Determine the form of g(x) = f [f(x)] & hence find the point of discontinuity of g , if any.

Sol. $f(x) = \begin{cases} 1+x & ; \quad 0 \le x \le 2 \\ 3-x & ; \quad 2 < x \le 3 \end{cases}$

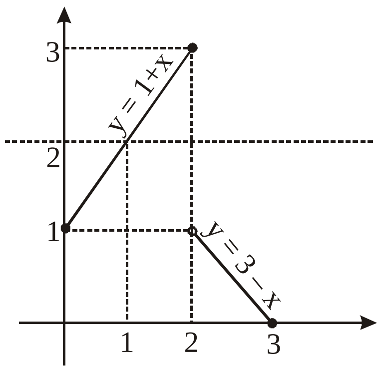


g(x) = f (f(x))

$$= \begin{cases} 1+f(x) & ; \quad 0 \le f(x) \le 2 \\ 3-f(x) & ; \quad 2 < f(x) \le 3 \end{cases}$$

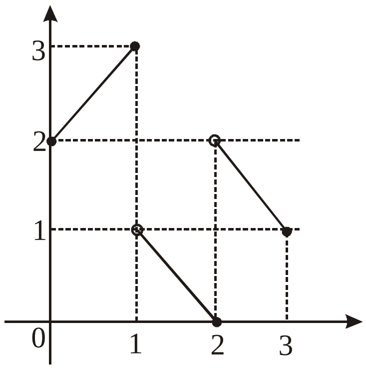
let f(x) = y

$$f(y) = \begin{cases} 1+y & ; \quad 0 \le y \le 2 \\ 3-y & ; \quad 2 < y \le 3 \end{cases}$$



$$= \begin{cases} 1+(1+x) & ; \quad 0 \le x \le 1 \\ 1+(3-x) & ; \quad 2 < x \le 3 \\ 3-(1+x) & ; \quad 1 < x \le 2 \end{cases}$$

$$= \begin{cases} 2+x & ; \quad 0 \le x \le 1 \\ 2-x & ; \quad 1 < x \le 2 \\ 4-x & ; \quad 2 \le x \le 3 \end{cases}$$



so the point of discontinuity

1, 2 **Ans**

Or

F.V.$|_x$ = LHL = RHL

6 Let [x] denote the greatest integer function & f(x) be defined in a neighbourhood of 2 by

$$f(x) = \begin{cases} \dfrac{\left(\exp\{(x+2)\ell n4\}\right)^{\frac{[x+1]}{4}} - 16}{4^x - 16} & , x<2 \\ A\dfrac{1-\cos(x-2)}{(x-2)\tan(x-2)} & , x>2 \end{cases}.$$

Find the values of A & f(2) in order that f(x) may be continuous at x = 2.

Sol. RHL$|_{x=2}$ = $\lim_{x \to 2^+} f(x)$

$$= \lim_{x \to 2^+} \frac{A(1-\cos(x-2))}{(x-2).\tan(x-2)}$$

put x = 2 + h

$$= \lim_{h \to 0} \frac{A(1-\cosh)}{h \tan h}$$

$$= \lim_{h \to 0} A\left(\frac{1-\cosh}{h^2}\right)\frac{1}{\left(\dfrac{\tan h}{h}\right)}$$

$$= \lim_{h \to 0} \frac{4^2.4^{\frac{-h}{2}} - 16}{4^2.4^{-h} - 16}$$

$$= \lim_{h \to 0} \frac{4^{-n/2} - 1}{4^{-h} - 1}$$

$$= \lim_{h \to 0} \left(\frac{4^{-h/2} - 1}{-\dfrac{h}{2}}\right).\left(-\frac{h}{2}\right)\frac{1}{\left(\dfrac{4^{-h} - 1}{-h}\right)(-h)}$$

$$= \ell n4.\ \frac{1}{2}.\frac{1}{\ell n4} = \frac{1}{2}$$

$RHL|_{x=2} = \frac{A}{2}$

since the function is contin.

$VF|_{x=2} = RHL|_{x=2} = LHL|_{x=2}$

$LHL|_{x=2} \Rightarrow \lim_{x \to 2^-} f(x)$

$V.F.|_{x=2} = \frac{A}{2} = \frac{1}{2}$

$$= \lim_{x \to 2^-} \frac{\left(e^{(x+2)\ell n4}\right) - 16^{\frac{[x+1]}{4}}}{4^x - 16}$$

$\boxed{V.F.|_{x=2} = \frac{1}{2}}$ **Ans**

$$= \lim_{x \to 2^-} \frac{4^{\frac{(x+2)([x]+1)}{4}} - 16}{4^x - 16}$$

$\boxed{A = 1}$ **Ans**

$$= \lim_{x \to 2^-} \frac{4^{\left(\frac{x+2}{2}\right)} - 16}{4^x - 16}$$

put $x = 2 - h$

$$= \lim_{x \to 0} \frac{4^{\frac{4-h}{2}} - 16}{4^{2-h} - 16}$$

7 The function $f(x) = \begin{cases} \left(\frac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}} & \text{if } 0 < x < \frac{\pi}{2} \\ b+2 & \text{if } x = \frac{\pi}{2} \\ (1+|\cos x|)^{\left(\frac{a|\tan x|}{b}\right)} & \text{if } \frac{\pi}{2} < x < \pi \end{cases}$

Determine the values of 'a' & 'b' , if f is continuous at $x = \pi/2$.

Sol. $V.F.|_{x=\frac{\pi}{2}} = b + 2$ ...(1)

$$LHL|_{x=\frac{\pi}{2}} = \lim_{x \to \frac{\pi-}{2}} f(x) = \lim_{x \to \left(\frac{\pi}{2}\right)^-} \left(\frac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}}$$

put $x = \frac{\pi}{2} - h$

$$LHL|_{x=\frac{\pi}{2}} = \lim_{h \to 0} \left(\frac{6}{5}\right)^{\frac{\tan(3\pi - 6h)}{\tan(5\pi/2 - 5h)}} = \lim_{h \to 0} \left(\frac{6}{5}\right)^{-\frac{\tan 6h}{\cot 5h}} = 1$$

$$RHL|_{x=\frac{\pi}{2}} = \lim_{x \to \left(\frac{\pi}{2}\right)^+} f(x)$$

$$= \lim_{x \to \left(\frac{\pi}{2}\right)^+} (1 - \cos x)^{-\frac{a}{b}\tan x}$$

put $x = \frac{\pi}{2} + h$

$= \lim_{h\to 0}(1+\sin h)^{\frac{a}{b}\cot h}$; 1^∞ form

$= e^{\lim_{h\to 0}(\sin h)\frac{a}{b}\cot h}$

$= e^{\lim_{h\to 0}\frac{a}{b}\cos h} = e^{\frac{a}{b}}$

since the function is conti so

$LHL|_{x=\frac{\pi}{2}} = RHL|_{x=\frac{\pi}{2}} = V.F.|_{x=\frac{\pi}{2}}$

$1 = e^{ab} = b + 2$

$\boxed{a = 0,\ b = -1}$

8 Let $f(x) = \begin{cases} \dfrac{\left(\frac{\pi}{2} - \sin^{-1}(1-\{x\}^2)\right)\sin^{-1}(1-\{x\})}{\sqrt{2}(\{x\}-\{x\}^3)} & ; \quad x \neq 0 \\ \dfrac{\pi}{2} & ; \quad x = 0 \end{cases}$

where $\{x\}$ is the fractional part of x. Consider another function g(x); such that

$g(x) = f(x)$; $x \geq 0$

$= 2\sqrt{2}f(x)$; $x < 0$

Discuss the continuity of the function f(x) & g(x) at $x = 0$.

Sol. $RHL|_{x=0} = \lim_{x\to 0^+} f(x)$

$$= \lim_{x\to 0^+} \frac{\left(\frac{\pi}{2} - \sin^{-1}(1-(x-[x]^2)\right)\sin^{-1}(1-x+[x])}{\sqrt{2}(x-[x]-(x-[x])^3)}$$

$$= \lim_{x\to 0^+} \frac{\left(\frac{\pi}{2} - \sin^{-1}(1-x^2)\right)\sin^{-1}(1-x)}{\sqrt{2}x(1-x^2)}$$

$$= \lim_{x\to 0^+} \frac{\cos^{-1}(1-x^2).\sin^{-1}(1-x)}{\sqrt{2}x.(1-x^2)}$$

$$= \frac{\pi}{2\sqrt{2}} \lim_{x\to 0^+} \frac{\cos^{-1}(1-x^2)}{x}$$

let $\cos^{-1}(1-x^2) = \theta$

$$1 - x^2 = \cos\theta$$

$$x^2 = 1 - \cos\theta$$

$$x = \sqrt{1-\cos\theta}$$

when $x \to 0^+$ then $\theta \to 0$

$$= \frac{\pi}{2\sqrt{2}} \lim_{\theta\to 0^+} \frac{\theta}{\sqrt{1-\cos\theta}}$$

$$= \frac{\pi}{2\sqrt{2}} \lim_{\theta\to 0^+} \frac{\theta}{\sqrt{2-\sin^2\theta/2}} = \frac{\pi}{4} \lim_{\theta\to 0^+} \frac{\theta}{|\sin\theta/2|}$$

$$\text{RHL}|_{x=0} = \frac{\pi}{4} \lim_{\theta\to 0^+} 2\left(\frac{\theta/2}{\sin\theta/2}\right) = \frac{\pi}{2}$$

$$\text{LHL}|_{x=0} = \lim_{x\to 0^-} f(x)$$

$$= \lim_{x\to 0^-} \frac{\left(\frac{\pi}{2} - \sin^{-1}(1-x-[x])^2\right)\sin^{-1}(1-x+[x])}{\sqrt{2}(x-[x]-(x-[x])^3)}$$

$$= \lim_{x\to 0^-} \frac{\frac{\pi}{2} - \sin^{-1}(1-(x+1)^2)\sin^{-1}(-x)}{\sqrt{2}(x+1-(x+1)^3)}$$

$$= \lim_{x\to 0^-} \frac{\left(\frac{\pi}{2}\sin^{-1}(-x^2-2x)\right)\sin^{-1}(-x)}{\sqrt{2}(x+1)(-x^2-2x)}$$

$$= \lim_{x\to 0^-} \frac{\cos^{-1}(-x^2-2x)\sin^{-1}(x)}{\sqrt{2}(x+1)(x^2+2x)}$$

$$= \lim_{x\to 0^-} \frac{\pi - \cos^{-1}(x^2+2x)}{\sqrt{2}(x+1)\ (x+2)} . \frac{\sin^{-1} x}{x}$$

$$\text{LHL}|_{x=0} = \frac{\pi}{4\sqrt{2}}$$

for f(x) since $\text{LHL}|_{x=0} \neq \text{RHL}|_{x=0}$ so the function is discontinuous at $x = 0$.

for $g(x) \Rightarrow$

$$\text{RHL}|_{x=0} = \lim_{x\to 0^+} g(x)$$

$$= \lim_{x\to 0^+} f(x) = \frac{\pi}{2}$$

$$\text{LHL}|_{x=0} = \lim_{x\to 0^-} 2\sqrt{2} f(x)$$

$$= 2\sqrt{2} \lim_{x\to 0^-} f(x)$$

$$= 2\sqrt{2}.\frac{\pi}{4\sqrt{2}} = \frac{\pi}{2}$$

$$g(0) = f(0) = \frac{\pi}{2}$$

9 If the function $f(x)$ defined as $f(x) = \begin{cases} -\frac{x^2}{2} & \text{for } x \le 0 \\ x^n \sin\frac{1}{x} & \text{for } x > 0 \end{cases}$ is continuous but not derivable at x = 0 then find the range of *n*.

Sol. $f(x) = \begin{cases} -\frac{x^2}{2} & \text{for } x \le 0 \\ x^n \sin\frac{1}{x} & \text{for } x > 0 \end{cases}$

f(x) is continuous at x = 0

f(0) = 0

$$L_1 = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(\frac{-x^2}{2}\right) = 0$$

$$L_2 = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^n \sin\frac{1}{x}$$

for continuous,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0^+} x^n \sin\left(\frac{1}{x}\right) = 0$$

limit is defined only when

$\therefore\ n > 0$

since f(x) is non-differentiable at x = 0

$$f'(0^-) = \lim_{h\to 0^-} \frac{f(h+0) - f(0)}{2} = \lim_{h\to 0^-} \frac{-\frac{h^2}{2} - 0}{h} = \lim_{h\to 0^-}\left(\frac{-h}{2}\right)$$

$$f'(0^+) = \lim_{h\to 0^+} \frac{f(h+0)-f(0)}{h} = \lim_{h\to 0^+} \frac{h^n \sin\frac{1}{h}}{h}$$

$f'(0^+) \neq f'(0^-)$

$$\Rightarrow \lim_{h\to 0^+} h^{n-1} \sin\left(\frac{1}{h}\right) \neq 0$$

only when $n - 1 \leq 0$

$\Rightarrow n \leq 1$ .....(ii)

from equation (i) & (ii)

$n \in (0, 1]$

10 . . . $f(0) = 0$ and $f'(0) = 1$. For a positive integer k, show that

$$\lim_{x\to 0} \frac{1}{x}\left(f(x) + f\left(\frac{x}{2}\right) + ...f\left(\frac{x}{k}\right)\right) = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{k}$$

Sol. $$\lim_{x\to 0} \frac{1}{x}\left[f(x) + f\left(\frac{x}{2}\right) + ... + f\left(\frac{x}{k}\right)\right]$$

$$= \lim_{x\to 0} \frac{f(x)}{x} + \frac{f\left(\frac{x}{2}\right)}{x} + ... + \frac{f\left(\frac{x}{k}\right)}{x}$$

$$= \lim_{x\to 0} \frac{f(x+0)-f(0)}{x} + \lim_{x\to 0} \frac{f\left(\frac{1}{x}+0\right) - f(0)}{\frac{x}{2}}\cdot\frac{1}{2} + ... + \lim_{x\to 0^+} \frac{f\left(\frac{x}{k}+0\right) - f(0)}{\frac{x}{k}}\cdot\frac{1}{k}$$

$$= f'(0) + \frac{1}{2} f'(0) + ... + \frac{1}{k} f'(0)$$

$$= 1 + \frac{1}{2} + ... + \frac{1}{k}$$

11 If f(x) $= \begin{cases} ax^2 - b & \text{if } |x| < 1 \\ -\frac{1}{|x|} & \text{if } |x| \geq 1 \end{cases}$ is derivable at x = 1. Find the values of a & b.

Sol. $$f(x) = \begin{cases} ax^2 - b & \text{if } |x| < 1 \\ -\frac{1}{|x|} & \text{if } |x| \geq 1 \end{cases}$$

f(x) is differentiable at x = 1, hence it is also continuous at x = 1

$\lim_{x \to 1} f(x) = f(1)$

$\Rightarrow \boxed{a - b = -1}$ ...(i)

$$f'(1) = \lim_{h \to 0^-} \frac{f(h+1) - f(1)}{h} = \lim_{h \to 0^-} \frac{a(h+1)^2 - b + 1}{h}$$

$$= \lim_{h \to 0^-} \frac{ah^2 + 2ah + a - b + 1}{h}$$

$$= \lim_{h \to 0^-} \frac{ah^2 + 2ah}{h} = \lim_{h \to 0^-} (ah + 2a) = 2a$$

$$f'(1^+) = \lim_{h \to 0^+} \frac{f(h+1) - f(1)}{h} = \lim_{h \to 0^+} \frac{\frac{-1}{|h+1|} + 1}{h}$$

$$= \lim_{h \to 0^+} \frac{\frac{-1+1+h}{1+h}}{h} = \lim_{h \to 0^+} \frac{1}{1+h} = 1$$

$f'(1^-) = f'(1^+)$

$\Rightarrow 2a = 1$

$a = 1/2$

$b = 3/2$

12 The function $f(x) = \begin{cases} ax(x-1) + b & \text{when } x < 1 \\ x - 1 & \text{when } 1 \le x \le 3 \\ px^2 + qx + 2 & \text{when } x > 3 \end{cases}$

Find the values of the constants a, b, p, q so that

(i) f(x) is continuous for all x (ii) f ' (1) does not exist (iii) f '(x) is continuous at x = 3

Sol. $f(x) = \begin{cases} ax(x-1) + b & \text{when } x < 1 \\ x - 1 & \text{when } 1 \le x \le 3 \\ px^2 + qx + 2 & \text{when } x > 3 \end{cases}$

f(x) is continous at x = 1

$\lim_{x \to 1} f(x) = f(1)$

$\Rightarrow \lim_{x \to 1^-} ax(x-1) + b = 0$

$\Rightarrow \boxed{b = 0 \ \& \ a \in R}$

$$f'(1) = \lim_{h\to 0} \frac{f(h+1)-f(1)}{h} = \begin{cases} \lim_{h\to 0^-} \dfrac{a(h+1)(h+1-1)+b}{h} \\ \lim_{h\to 0^+} \dfrac{h+1-1}{h} \end{cases}$$

$$= \begin{cases} \lim_{h\to 0^-} \dfrac{a(h+1)h+0}{h} \\ \lim_{h\to 0^+} \dfrac{h}{h} \end{cases} = \begin{cases} \lim_{h\to 0^-} a(h+1) \\ 1 \end{cases}$$

$$= \begin{cases} a \\ 1 \end{cases}$$

$\because$ $f'(1) = \text{DNE} \Rightarrow a \neq 1$

$\therefore$ $a \in R - \{1\}$ & $b = 0$

f(x) is cont. at x = 3

$$\lim_{x\to 3} f(x) = f(3)$$

$$\Rightarrow \lim_{x\to 3} (px^2 + 9x + 2) = 2$$

$\Rightarrow 9p + 3q + 2 = 2$

$\Rightarrow 9p + 3q = 0$ ....(i)

$\because$ f '(x) is cont. at x = 3, hence f(x) is diff. at x = 3

$$f'(3) = \lim_{h\to 0} \frac{f(h+3)-f(3)}{h} = \begin{cases} \lim_{h\to 0^-} \dfrac{3+h-1-2}{h} \\ \lim_{h\to 0^+} \dfrac{p(h+3)^2+q(h+3)+2-2}{h} \end{cases}$$

$$= \begin{cases} \lim_{h\to 0^-} \dfrac{h}{n} \\ \lim_{h\to 0^+} \dfrac{ph^2+6ph+qh+9p+3q}{h} \end{cases} = \begin{cases} 1 \\ \lim_{h\to 0^+} \dfrac{ph^2+6ph+qh}{h} \end{cases}$$

[from equation (i) 9p + 3q = 0]

$$= \begin{cases} 1 \\ \lim_{h\to 0^+} (ph+6p+q) \end{cases} = \begin{cases} 1 \\ 6p+q \end{cases}$$

$\therefore$ $f'(3^+) = f'(3^-) \Rightarrow 6p + q = 0$ ...(ii)

solving equation (i) & (ii) p = 1/3, q = – 1

$a \in R - \{1\}$, b = 0, p = 1/3 , q = – 1

13 Discuss the continuity on $0 \leq x \leq 1$ & differentiability at x = 0 for the function.

$f(x) = x \sin\frac{1}{x} . \sin\frac{1}{x \sin\frac{1}{x}}$ where $x \neq 0$, $x \neq 1/r\pi$ & $f(0) = f(1/r\pi) = 0$,

$r = 1, 2, 3, \ldots\ldots\ldots$

Sol. $f(x) = x . \sin\frac{1}{x} . \sin\frac{1}{x.\sin\frac{1}{x}}$ $x \neq 0, 1/r\pi$

$f(0) = 0 = f\left(\frac{1}{r\pi}\right)$, $r = 1, 2, 3....$

$$f'(0) = \lim_{h\to 0}\frac{f(h+0) - f(0)}{h}$$

$$= \lim_{h\to 0}\frac{h\sin\left(\frac{1}{h}\right).\sin\left(\frac{1}{h\sin\left(\frac{1}{h}\right)}\right) - 0}{h}$$

$$= \lim_{h\to 0}\frac{h\sin\left(\frac{1}{h}\right).\sin\left(\frac{1}{h\sin\left(\frac{1}{h}\right)}\right)}{h}$$

$$= \lim_{h\to 0}\sin\left(\frac{1}{h}\right).\sin\left(\frac{1}{h\sin\left(\frac{1}{h}\right)}\right)$$

$$= \lim_{h\to 0}\ \underbrace{\sin\left(\frac{1}{h}\right)}_{1-\le\ \ \le 1}\ \underbrace{\sin\left(\frac{1}{h\sin(1/h)}\right)}_{-1\le\ \ \le 1}$$

$= \text{DNE}$

so f(x) is not differentiable at $x = 0$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin\left(\frac{1}{x}\right).\sin\left(\frac{1}{x\sin(1/x)}\right)$$

$$= \lim_{x\to 0} \underbrace{x}_{\to 0} .\underbrace{\sin(1/x)}_{-1\le\ \le 1} . \underbrace{\sin\left(\frac{1}{n\sin(1/x)}\right)}_{-1\le\ \le 1}$$

$$= 0$$

$$= f(0)$$

$$\lim_{x\to\frac{1}{r\pi}} f(x) = \lim_{x\to\frac{1}{\pi r}} x\sin\left(\frac{1}{x}\right).\sin\left(\frac{1}{x\sin\left(\frac{1}{x}\right)}\right)$$

$$= \lim_{x\to\frac{1}{r\pi}} x.\sin\left(\frac{1}{x}\right).\sin\left(\frac{\frac{1}{\sin\left(\frac{1}{x}\right)}}{\left(\frac{1}{x}\right)}\right)$$

$$= \lim_{x\to\frac{1}{r\pi}} \underbrace{x.\sin\left(\frac{1}{x}\right)}_{\to 0}.\underbrace{\sin\left(\frac{1}{x\sin\left(\frac{1}{x}\right)}\right)}_{-1\le\ \ \le 1}$$

$$= 0$$

$$= f\left(\frac{1}{r\pi}\right)$$

Hence function is continuous $\forall\ x \in [0, 1]$

14 $f(x) = \begin{cases} 1-x & , (0\le x\le 1) \\ x+2 & , (1<x<2) \\ 4-x & , (2\le x\le 4) \end{cases}$ Discuss the continuity & differentiability of $y = f[f(x)]$ for $0 \le x \le 4$.

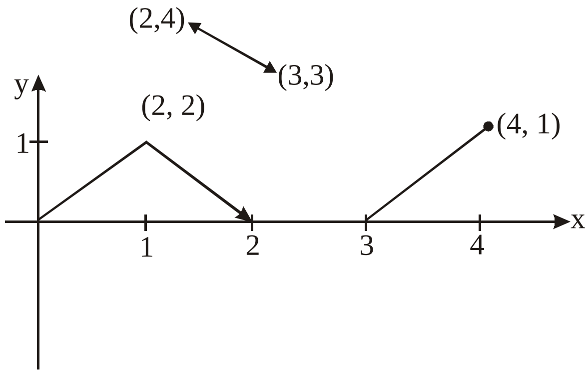
Sol. $f(x) = \begin{cases} 1-x & , (0 \le x \le 1) \\ x+2 & , (1 < x < 2) \\ 4-x & , (2 \le x \le 4) \end{cases}$

$$f(f(x)) = \begin{cases} 1 - f(x) & ; \quad 0 \le f(x) \le 1 \\ f(x) + 2 & ; \quad 1 < f(x) < 2 \\ 4 - f(x) & ; \quad 2 \le f(x) \le 4 \end{cases}$$

$$= \begin{cases} 1 - 1 - x & ; \quad 0 \le x \le 1 \cap 1 \le 1 - x \le 1 \Rightarrow 0 \le x \le 1 \\ 1 - x - 2 & ; \quad 1 < x < 2 \cap 0 \le x + 2 \le 1 \Rightarrow -2 \le x \le -1 \\ 1 - 4 + x & ; \quad 2 \le x \le 4 \cap 0 \le 4 - x \le 1 \Rightarrow 3 \le x \le 4 \\ 1 - x + 2 & ; \quad 0 \le x \le 1 \cap 1 < 1 - x < 2 \Rightarrow -1 < X < 0 \\ x + 2 + 2 & ; \quad 1 < x < 2 \cap 1 < x + 2 < 2 \Rightarrow -1 < x < 0 \\ 4 - x + 2 & ; \quad 2 \le x \le 4 \cap 1 < 4x < 2 \Rightarrow 2 < x < 3 \\ 4 - 1 + x & ; \quad 0 \le x \le 1 \cap 2 \le 1 - x \le 4 \Rightarrow -3 \le x \le -1 \\ 4 - x - 2 & ; \quad 1 < x < 2 \cap 2 \le 4 \Rightarrow 0 \le x \le 2 \\ 4 - 4 + x & ; \quad 2 \le x \le 4 \cap 2 \le 4 - x \le 4 \Rightarrow 0 \le x \le 2 \end{cases}$$

$$= \begin{cases} x & ; \quad 0 \le x \le 1 \\ x - 3 & ; \quad 3 \le x \le 4 \\ -x + 6 & ; \quad 2 < x < 3 \\ -x + 2 & ; \quad 1 < x < 2 \end{cases}$$

$$f(f(x)) = \begin{cases} x & ; \quad 0 \le x \le 1 \\ -x - 2 & ; \quad 1 < x < 2 \\ x & ; \quad x = 2 \\ -x + 6 & ; \quad 2 < x < 3 \\ x - 3 & ; \quad 3 \le x \le 4 \end{cases}$$



$\therefore$ f(x) is continuous at x = 1 & discunt.

at x = 2, 3 & non diff. at x = 1, 2, 3

15 Let f be a function that is differentiable every where and that has the following properties:
(i) f (x + h) = f (x) · f (h)   (ii) f (x) > 0 for all real x.   (iii)   f ’(0) = – 1

Use the definition of derivative to find $f'(x)$ in terms of $f(x)$.

Sol. $f(x+h) = f(x) \,.\, f(h)$ $\left|\begin{array}{l} x=0 \\ h=0 \end{array}\right.$ $f(0)\,(f(0)-1) = 0 \Rightarrow f(0) = 1$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x).f(h)-f(x)}{h} = \lim_{h\to 0} \frac{f(h)-1}{h} f(x)$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{f(h)-f(0)}{h}\ f(x)$$

$$= f'(0)\ f(x)$$

$$\Rightarrow f'(x) = -f(x)$$

$$\therefore f'(x) = -f(x)$$

16 Discuss the continuity & the derivability of 'f' where f (x) = degree of $(u^{x^2} + u^2 + 2u - 3)$ at $x = \sqrt{2}$.

Sol. $f(x) =$ degree of $(u^{x^2} + u^2 + 2u - 3)$ at $x = \sqrt{2}$

$$= \begin{cases} 2 & ; \quad x \le \sqrt{2} \\ x^2 & ; \quad x > \sqrt{2} \end{cases}$$

$$f'(\sqrt{2}) = \lim_{h\to o} \frac{f(h+\sqrt{2}) - f(\sqrt{2})}{h}$$

$$= \begin{cases} \lim\limits_{h\to o^+} \dfrac{f(h+\sqrt{2}) - f(\sqrt{2})}{h} \\ \lim\limits_{h\to o^-} \dfrac{f(h+\sqrt{2}) - f(\sqrt{2})}{h} \end{cases}$$

$$= \begin{cases} \lim\limits_{h\to o^+} \dfrac{(h+\sqrt{2})^2 - 2}{h} \\ \lim\limits_{h\to o^-} \dfrac{2-2}{h} \end{cases}$$

$$= \begin{cases} \lim\limits_{h\to o^+} \dfrac{2+2\sqrt{2}h + h^2 - 2}{h} \\ 0 \end{cases}$$

$$= \begin{cases} \lim\limits_{h\to o^+} \dfrac{h^2 + 2\sqrt{2}h}{h} \\ 0 \end{cases}$$

$$= \begin{cases} \lim\limits_{h\to o^+} (h + 2\sqrt{2}) \\ 0 \end{cases}$$

$$= \begin{cases} 2\sqrt{2} \\ 0 \end{cases}$$

$$\therefore \; f'(\sqrt{2}^-) \neq f'(\sqrt{2}^+)$$

Hance f(x) is non differentiable at $x = \sqrt{2}$

$$\lim_{x\to\sqrt{2}} f(x) = \lim_{x\to\sqrt{2}} x^2$$

$$= 2$$

$$= f(\sqrt{2})$$

$$\Rightarrow \; f(\sqrt{2}) = \lim_{x\to\sqrt{2}} f(x)$$

Hance f(x) is confinous at $x = \sqrt{2}$

17 Let f (x) be a function defined on (–a, a) with a > 0. Assume that f (x) is continuous at x = 0 and $\lim\limits_{x\to 0} \dfrac{f(x) - f(kx)}{x} = \alpha$, where $k \in (0, 1)$ then compute $f'(0^+)$ and $f'(0^-)$, and comment upon the differentiability of f at x = 0.

Sol. $\because \lim\limits_{x\to 0} \dfrac{f(x) - f(k\alpha)}{x} = \alpha$

$$\Rightarrow \lim_{x\to o} \frac{f(x) - f(0) + f(0) - (kx)}{x} = \alpha$$

$$\Rightarrow \lim_{x\to o} \frac{f(x) - f(0) - f(kx) + f(0)}{x} = \alpha$$

$$\Rightarrow \lim_{x\to o} \left(\frac{f(x) - f(0)}{x} - \frac{f(kx) - f(0)}{x} \right) = \alpha$$

$$\Rightarrow \left(\lim_{x\to 0} \frac{f(x) - f(0)}{x} \right) - \left(\lim_{x\to 0} \frac{f(kx) - f(0)}{kx} \right) k = \alpha$$

$$\Rightarrow \begin{cases} \lim\limits_{x\to 0^-}\dfrac{f(x)-f(0)}{x} - \lim\limits_{x\to 0^-}\dfrac{f(kx)-f(0)}{kx}.k = \alpha \\ \lim\limits_{x\to 0^+}\dfrac{f(x)-f(0)}{x} - \lim\limits_{x\to 0^+}\dfrac{f(kx)-f(0)}{kx}.k = \alpha \end{cases}$$

$$= \begin{cases} f'(0^-) - kf'(0^-) = \alpha \\ f'(0^+) - kf'(0^+) = \alpha \end{cases}$$

$$= \begin{cases} (1-k)f'(0^-) = \alpha \\ (1-k)f'(0^+) = \alpha \end{cases}$$

$$= \begin{cases} f'(0^-) = \dfrac{\alpha}{1-k} \\ f'(0^+) = \dfrac{\alpha}{1-k} \end{cases}$$

$$\therefore\ f'(0) = f'(0^-) = f'(0^+) = \frac{\alpha}{1-k}$$

18 A derivable function $f: R^+ \to R$ satisfies the condition $f(x) - f(y) \geq \ln(x/y) + x - y$ for every $x, y \in R^+$. If g denotes the derivative of f then compute the value of the sum $\sum\limits_{n=1}^{100} g\left(\frac{1}{n}\right)$.

Sol. $f(x) - f(x) \geq \ell n(x/y) + x - y$

$\Rightarrow f(x) - f(y) \geq \ell nx - \ell ny + x - y$

$$\Rightarrow \frac{f(x)-f(y)}{x-y} \geq \frac{\ell n x - my}{x-y} + 1 \qquad [\text{for } x \neq y]$$

$$\Rightarrow \lim_{x\to y}\frac{f(x)-t(y)}{x-y} \geq \lim_{x\to y}\frac{\ell n\ x - \ell ny}{x-y} + 1$$

$$\Rightarrow \lim_{h\to 0}\frac{f(h+y)-f(y)}{h} \geq \lim_{h\to 0}\frac{m\left(\frac{y+h}{y}\right)}{h} + 1$$

$$\Rightarrow f'(y) \geq \lim_{h\to 0} \ell n\left(1+\frac{h}{y}\right)^{1/h} + 1$$

19 If $y = \dfrac{x^2}{2} + \dfrac{1}{2}x\sqrt{x^2+1} + \ln\sqrt{x+\sqrt{x^2+1}}$ prove that $2y = xy' + \ln y'$. where ' denotes the derivative.

[Sol. $y = \dfrac{x^2}{2} + \dfrac{1}{2}x\sqrt{x^2+1} + \ln\sqrt{x+\sqrt{x^2+1}}$

$$y' = x + \frac{1}{2}\left[\frac{x^2}{\sqrt{x^2+1}} + \sqrt{x^2+1}\right] + \frac{1}{2\sqrt{x^2+1}}$$

$$= x + \frac{1}{2}\left[\frac{2x^2+1}{\sqrt{x^2+1}}\right] + \frac{1}{2\sqrt{x^2+1}}$$

$$= x + \frac{1}{2\sqrt{x^2+1}}[2(x^2+1)]$$

$$y' = x + \sqrt{x^2+1}$$

Also $2y = x^2 + x\sqrt{x^2+1} + ln(x + \sqrt{x^2+1})$

$= x(x + \sqrt{x^2+1}) + ln(x + \sqrt{x^2+1}) = xy' + ln\, y'$ Hence proved]

20 If $y = \sec 4x$ and $x = \tan^{-1}(t)$, prove that $\frac{dy}{dt} = \frac{16t(1-t^4)}{(1-6t^2+t^4)^2}$.

[Sol. $y = \frac{1}{\cos 4x} = \frac{1+\tan^2 2x}{1-\tan^2 2x}$ ....(1)

using $\tan x = t$ (given)

$\tan 2x = \frac{2t}{1-t^2}$

substituting in (1)

$$y = \frac{1+\frac{4t^2}{(1-t^2)^2}}{1-\frac{4t^2}{(1-t^2)^2}} = \frac{(1+t^2)^2}{(1-t^2)^2-4t^2} = \frac{(1+t^2)^2}{1-6t^2+t^4}$$

$$\frac{dy}{dt} = \frac{(1-6t^2+t^4)\cdot 2(1+t^2)\cdot 2t - (1+t^2)(4t^3-12t)}{(1-6t^2+t^4)^2}$$

$$= \frac{4t(1+t^2)[(1-6t^2+t^4)-(1+t^2)(t^2-3)]}{\left(1-(t^2+t^4)^2\right)} = \frac{4t(1+t^2)(1-t^2)}{(1-6t^2+t^4)^2} = \frac{4t(1-t^4)}{(1-6t^2+t^4)^2}$$]

21 If $x = \frac{1+lnt}{t^2}$ and $y = \frac{3+2lnt}{t}$. Show that $y\frac{dy}{dx} = 2x\left(\frac{dy}{dx}\right)^2 + 1$.

[Sol. $\frac{dx}{dt} = \frac{t-(1+lnt)2t}{t^4} = \frac{t(1-2-lnt)}{t^4} = -\frac{(1+2lnt)}{t^3}$

$$\frac{dy}{dt} = \frac{t\left(\frac{2}{t}\right)-(3+2\ln t)}{t^2} = -\frac{(1+2\ln t)}{t^2}$$

$$\frac{dy}{dx} = \frac{1+2\,ln\,t}{t^2}.\frac{t^3}{1+2\,ln\,t} = t$$

Now L.H.S. $= \frac{3+2lnt}{t}.t = 3 + 2\,lnt$

$$\text{R.H.S.} = \frac{2(1+\ln t)}{t^2} \, . \, t^2 + 1 = 3 + 2\ln 2$$

$\Rightarrow$ L.H.S. = R.H.S.]

22 If $y = 1 + \frac{x_1}{x - x_1} + \frac{x_2 \, . \, x}{(x-x_1)(x-x_2)} + \frac{x_3 \, . x^2}{(x-x_1)(x-x_2)(x-x_3)} + ...$ upto (n+1) terms then prove that

$$\frac{dy}{dx} = \frac{y}{x}\left[\frac{x_1}{x_1 - x} + \frac{x_2}{x_2 - x} + \frac{x_3}{x_3 - x} + ... + \frac{x_n}{x_n - x}\right]$$

[Sol. adding term by term

$$y = \frac{x^n}{(x-x_1)(x-x_2)\,(x-x_3)\,....(x-x_n)}$$

$$y = \frac{x}{(x-x_1)} \, . \, \frac{x}{(x-x_2)} \, . \, \frac{x}{(x-x_3)} \, ... \frac{x}{(x-x_n)}$$

$$\ln y = \ln \frac{x}{(x-x_1)} + \ln \frac{x}{(x-x_2)} + \ln \frac{x}{(x-x_3)} + ... + \ln \frac{x}{(x-x_n)}$$

$$\text{now } D\left(\frac{x}{x-x_n}\right) = \frac{x-x_n}{x}\left(\frac{(x-x_n)-x}{(x-x_n)^2}\right) = \frac{1}{x}\left(\frac{x_n}{x_n - x}\right)$$

$$\text{Hence } \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\left[\frac{x_1}{x_1 - x} + \frac{x_2}{x_2 - x} + ... + \frac{x_n}{x_n - x}\right]$$

$$\frac{dy}{dx} = \frac{y}{x}\left[\frac{x_1}{x_1 - x} + \frac{x_2}{x_2 - x} + ... + \frac{x_n}{x_n - x}\right]$$]

23 Suppose $f(x) = \tan\left(\sin^{-1}(2x)\right)$

(a) Find the domain and range of f.

(b) Express f (x) as an algebraic function of x.

(c) Find $f'(1/4)$. [Ans. (a) $\left(-\frac{1}{2}, \frac{1}{2}\right)$, $(-\infty, \infty)$; (b) $f(x) = \frac{2x}{\sqrt{1-4x^2}}$; (c) $\frac{16\sqrt{3}}{9}$]

[Sol. $f(x) = \tan\left(\sin^{-1}(2x)\right)$

(a) for f to be well defined

$$-1 < 2x < 1 \quad \Rightarrow \quad -\frac{1}{2} < x < \frac{1}{2} \quad [\because \text{ for } x = \pm\frac{1}{2}, \tan\frac{\pi}{2} \text{ is not defined}]$$

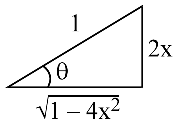
Hence domain is $\left(-\frac{1}{2}, \frac{1}{2}\right)$

for $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, $\sin^{-1}2x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ hence for $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ can take all real values.

Hence range of f is $x \in R$

(b) $f(x) = \tan\theta$ where $\sin^{-1}(2x) = \theta \Rightarrow \sin\theta = 2x$

$$f(x) = \frac{2x}{\sqrt{1-4x^2}}$$



(c) $f'(x) = \frac{\sec^2\left(\sin^{-1}(2x)\right)}{\sqrt{1-4x^2}}\cdot 2$

$$f'\left(\frac{1}{4}\right) = \frac{2\sec^2\left(\sin^{-1}\left(\frac{1}{2}\right)\right)}{\sqrt{1-\frac{1}{4}}} = \frac{2\times 2}{\sqrt{3}}\cdot\frac{4}{3} = \frac{16}{3\sqrt{3}} = \frac{16\sqrt{3}}{9}$$]

24 If $x = \tan\frac{y}{2} - \ln\left[\frac{\left(1+\tan\frac{y}{2}\right)^2}{\tan\frac{y}{2}}\right]$. Show that $\frac{dy}{dx} = \frac{1}{2}\sin y(1+\sin y+\cos y)$.

Sol Put $\tan\frac{y}{2} = t$ $\quad\therefore\quad$ $\sin y = \frac{2t}{1+t^2}, \cos y = \frac{1-t^2}{1+t^2}$

$\therefore \quad 1+\sin y+\cos y = \frac{2+2t}{1+t^2}$

and $\quad y = 2\tan^{-1} t \qquad$ ...(1)

$\therefore \quad \frac{dy}{dt} = \frac{2}{1+t^2} \qquad$ ...(2)

Now $\quad x = t - 2\log(1+t) + \log t$

$\therefore \quad \frac{dx}{dt} = 1 - \frac{2}{1+t} + \frac{1}{t} = \frac{t^2+1}{t(t+1)}$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} + \frac{dx}{dt} = \frac{2}{1+t^2}\cdot\frac{t^2+t}{1+t^2}$, by (2) & (3)

or $\quad \frac{dy}{dx} = \frac{2t}{1+t^2}\cdot\frac{1}{2}\,\frac{2t+2}{1+t^2}$

$$= \frac{1}{2}\sin y(1+\sin y+\cos y),$$ by (1)

25 If $y = \arccos\sqrt{\frac{\cos 3x}{\cos^3 x}}$ then show that $\frac{dy}{dx} = \sqrt{\frac{6}{\cos 2x + \cos 4x}}$, $\sin x > 0$.

Sol We have,

$$y = \cos^{-1}\sqrt{\frac{\cos 3x}{\cos^3 x}}$$

$\therefore \quad \cos y = \sqrt{\frac{\cos 3x}{\cos^3 x}}$

$\Rightarrow \quad \cos y = \sqrt{\frac{4\cos^3 x - 3\cos x}{\cos^3 x}}$

$\Rightarrow \quad \cos y = \sqrt{4 - 3\sec^2 x}$

$$\Rightarrow \quad \cos^2 y = 4 - 3\left(1 + \tan^2 x\right)$$

$$\Rightarrow \quad 1 - \cos^2 y = 3\tan^2 x$$

$$\Rightarrow \quad \sin^2 y = 3\tan^2 x$$

$$\Rightarrow \quad \sin y = \sqrt{3}\tan x$$

Differentiating both side with respect to x, we get, $\cos y \frac{dy}{dx} = \sqrt{3}\sec^2 x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{3}}{\cos y \cos^2 x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{3}}{\cos^2 x}\sqrt{\frac{\cos^3 x}{\cos 3x}} = \sqrt{\frac{3}{\cos x \cos 3x}}$$

Hence Proved $\frac{dy}{dx} = \sqrt{\frac{6}{\cos 2x + \cos 4x}}$, $\sin x > 0$.

26 Prove that if $|a_1 \sin x + a_2 \sin 2x + ... + a_n \sin nx| \le |\sin x|$ for $x \in R$, then $|a_1 + 2a_1 + 3a_3 + ... + na_n| \le 1$

[Sol. Let $f(x) = a_1 \sin x + a_2 \sin 2x + ... + a_n \sin nx$

$f'(x) = a_1 \cos x + 2a_2 \cos 2x + ... + na_n \cos nx$

$f'(0) = a_1 + 2a_2 + ... + na_n$

Hence TPT $|f'(0)| \le 1$

Given $|f(x)| \le |\sin x|$ for $x \in R$

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

$$f'(0) = \lim_{h\to 0} \frac{f(h)}{h} \qquad \text{(as } f(0) = 0\text{)}$$

$$|f'(0)| = \lim_{h\to 0}\left|\frac{f(h)}{h}\right| \le \lim_{h\to 0}\left|\frac{\sin h}{h}\right| = 1 \qquad [\text{as } |f(x)| \le |\sin x|\,]$$

Hence $|f'(0)| \le 1$]

27 Show that the substitution $z = ln\left(\tan\frac{x}{2}\right)$ changes the equation $\frac{d^2y}{dx^2} + \cot x\frac{dy}{dx} + 4y\,\mathrm{cosec}^2 x = 0$ to $(d^2y/dz^2) + 4y = 0$.

Sol Since $x = l\,n\tan\left(\frac{x}{2}\right)$

$$\therefore \quad \frac{dz}{dx} = \mathrm{cosec}\,x \quad \text{or} \quad \frac{dx}{dz} = \sin x \qquad ...(1)$$

$$\text{Now,} \quad \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \mathrm{cosec}\,x\cdot\frac{dy}{dz} \qquad \text{[From (1)]} \qquad ...(2)$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\mathrm{cosec}\,x\,\frac{dy}{dz}\right)$$

$$= \operatorname{cosec} x \frac{d}{dx}\left(\frac{dy}{dz}\right) + \frac{dy}{dz}\left(-\operatorname{cosec} x \cot x\right)$$

$$= \operatorname{cosec} x \cdot \frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx} - \operatorname{cosec} x \cot x \frac{dy}{dz}$$

$$= \operatorname{cosec}^2 x \frac{d^2y}{dz^2} - \operatorname{cosec} x \cot x \frac{dy}{dz} \qquad \text{[From (1)]} \qquad ....(3)$$

But given $\frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0$

$$\operatorname{cosec}^2 x \frac{d^2y}{dz^2} - \operatorname{cosec} x \cot x \frac{dy}{dz} + \cot x \operatorname{cosec} x \frac{dy}{dz} + 4y \operatorname{cosec}^2 x = 0 \qquad \text{[From (2) and (3)]}$$

$$\Rightarrow \quad \operatorname{cosec}^2 x \frac{d^2y}{dz^2} + 4y \operatorname{cosec}^2 x = 0 \quad \text{or} \quad \frac{d^2y}{dz^2} + 4y = 0$$

28 Let $f(x) = \begin{cases} xe^x & x \le 0 \\ x + x^2 - x^3 & x > 0 \end{cases}$ then prove that

(a) f is continuous and differentiable for all x. (b) $f\,'$ is continuous and differentiable for all x.

[Sol. $f'(x) = \begin{cases} xe^x + e^x = e^x(x+1), & x < 0 \\ 1 + 2x - 3x^2 & x > 0 \end{cases}$

$\lim_{x\to 0^-} f'(x) = 1; \quad \lim_{x\to 0^+} f'(x) = 1$

hence f (x) is continuous hence f is continuous and differentiable at x = 0

Again $f''(x) = \begin{cases} e^x + (x+1)e^x = e^x(x+2), & x < 0 \\ 2 - 6x & x > 0 \end{cases}$

$\lim_{x\to 0^+} f''(x) = \lim_{x\to 0^-} f''(x) = 2 \quad \Rightarrow \quad f'(x)$ is also continuous and differentiable]

29 Let $f(x) = \begin{vmatrix} a+x & b+x & c+x \\ \ell+x & m+x & n+x \\ p+x & q+x & r+x \end{vmatrix}$. Show that $f''(x) = 0$ and that $f(x) = f(0) + kx$ where k denotes the sum of all the co-factors of the elements in f(0).

[Hint: $f'(x) = \begin{vmatrix} 1 & 1 & 1 \\ \ell+x & m+x & n+x \\ p+x & q+x & r+x \end{vmatrix} + \begin{vmatrix} a+x & b+x & c+x \\ 1 & 1 & 1 \\ p+x & q+x & r+x \end{vmatrix} + \begin{vmatrix} a+x & b+x & c+x \\ \ell+x & m+x & n+x \\ 1 & 1 & 1 \end{vmatrix}$

$f''(x) = 0$ (obviously – two identical rows)

$f'(x) = k \Rightarrow f(x) = kx + x, \quad f(0) = c$

$\Rightarrow f(x) = f(0) + kx$. Note that $f'(x) = k$

$$\Rightarrow f'(0) = k = \begin{vmatrix} 1 & 1 & 1 \\ \ell & m & n \\ p & q & r \end{vmatrix} + \begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ p & q & r \end{vmatrix} + \begin{vmatrix} a & b & c \\ \ell & m & n \\ 1 & 1 & 1 \end{vmatrix}$$

$= (c_{11} + c_{12} + c_{13}) + (c_{21} + c_{22} + c_{23}) + (c_{31} + c_{32} + c_{33})$

= sum of co-factors of elements f(0)]

30 If Y = sX and Z = tX, where all the letters denotes the functions of x and suffixes denotes the differentiation w.r.t. x then prove that $\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = X^3 \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}$

Sol Since $Y = sX$ and $Z = tX$ ...(1)

$\therefore$ $Y_1 = sX_1 + Xs_1$ and $Z_1 = tX_1 + Xt_1$ ...(2)

$\Rightarrow$ $Y_2 = sX_2 + Xs_2 + 2s_1X_1$ and $Z_2 = tX_2 + Xt_2 + 2t_1X_1$ ...(3)

$$L.H.S = \begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}$$

$$\begin{vmatrix} X & sX & tX \\ X_1 & sX_1 + Xs_1 & tX_1 + Xt_1 \\ X_2 & sX_2 + Xs_2 + 2s_1X_1 & tX_2 + Xt_2 + 2t_1X_1 \end{vmatrix}$$ [From (1),(2) and (3)]

Applying $C_2 \to C_2 - sC_1$ and $C_3 \to C_3 - tC_1$

$$= \begin{vmatrix} X & 0 & 0 \\ X_1 & Xs_1 & Xt_1 \\ X_2 & Xs_2 + 2s_1X_1 & Xt_2 + 2t_1X_1 \end{vmatrix}$$

Expand w.r.t. first row, then

$$= X \begin{vmatrix} Xs_1 & Xt_1 \\ Xs_2 + 2s_1X_1 & Xt_2 + 2t_1X_1 \end{vmatrix}$$

$$= X^3 \begin{vmatrix} s_1 & t_1 \\ Xs_2 + 2s_1X_1 & Xt_2 + 2t_1X_1 \end{vmatrix}$$

Applying $R_2 \to R_2 - 2X_1R_1$ $= X^2 \begin{vmatrix} s_1 & t_1 \\ Xs_2 & Xt_2 \end{vmatrix} = X^3 \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix} = R.H.S.$

28 A function $f : R \to R$ is defined as $f(x) = \lim\limits_{n\to\infty} \dfrac{ax^2 + bx + c + e^{nx}}{1 + c.e^{nx}}$ where f is continuous on R. Find the value of a, band c.

Sol. $f(x) = \lim\limits_{n\to\infty} \dfrac{ax^2 + bx + c + e^{nx}}{1 + c.e^{nx}}$

$$= \begin{cases} \lim\limits_{n\to\infty} \dfrac{ax^2 + bx + c + e^{nx}}{1 + c.e^{nx}} & ; x < 0 \\ \lim\limits_{n\to\infty} \dfrac{ax^2 + bx + c + e^{nx}}{1 + c.e^{xn}} & ; x = 0 \\ \lim\limits_{n\to\infty} \dfrac{ax^2 + bx + c + e^{nx}}{1 + c.e^{xn}} & ; x > 0 \end{cases}$$

$$= \begin{cases} \dfrac{ax^2+bx+c+0}{1+c.0} & ; \quad x<0\left(\lim\limits_{n\to\infty} e^{nx}=0\right) \\ \dfrac{c+1}{c+1} & ; \quad x=0 \\ \lim\limits_{n\to\infty} \dfrac{\dfrac{ax^2}{e^{nx}}+\dfrac{bx}{e^{nx}}+\dfrac{c}{e^{nx}}+1}{\dfrac{1+c}{e^{nx}}} & ; \quad x>0 \end{cases}$$

$$\left(\lim_{h\to\infty} e^{hx}=\infty\right)$$

$$= \begin{cases} ax^2+bx+c & ; \quad x<0 \\ 1 & ; \quad x=0 \\ \dfrac{1}{c} & ; \quad x>0 \end{cases}$$

since f(x) is continuous function $\forall\ x \in R$

$$\therefore \lim_{x\to0^+} f(x) = \lim_{x\to0^-} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to0^+}\left(\frac{1}{c}\right) = \lim_{x\to0^-}(ax^2+bx+c) = 1 \qquad \Rightarrow \lim_{x\to0^+}\frac{1}{c} = 1 \ \& \ \lim_{x\to0^-}(ax^2+bx+c)=1$$

$$\Rightarrow \frac{1}{c}=1 \qquad\qquad \Rightarrow a - 0 + 3.0 + c = 1$$

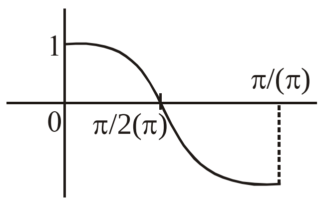
$\therefore$ c = 1 $\qquad\qquad \Rightarrow$ c = 1

$\therefore$ c = 1, a, b $\in$ R

29 Discuss the continuity of f in [0,2] where $f(x) = \begin{cases} |4x-5|\,[x] & \text{for } x>1 \\ [\cos \pi x] & \text{for } x \le 1 \end{cases}$; where [x] is the greatest integer not greater than x.

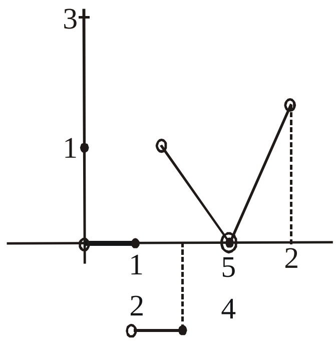
Sol. f(x) = cos πx

$$[\cos \pi x] = \begin{cases} 1 & ; \quad x=0 \\ 0 & ; \quad x<x\le \dfrac{1}{2} \\ -1 & ; \quad \dfrac{1}{2}<x\le 1 \end{cases}$$



$$|4x-5|\,[x] = \begin{cases} |4x-5|\ ; & 1 < x < 2 \\ 6\ ; & x = 2 \end{cases} = \begin{cases} (4x-5) & ; \quad 1 < x < \frac{5}{4} \\ 4x-5 & ; \quad \frac{5}{4} \le x < z \\ 6 & ; \quad x = 2 \end{cases}$$

$$f(x) = \begin{cases} 1 & ; \quad x = 0 \\ 0 & ; \quad 0 < x \le \frac{1}{2} \\ -1 & ; \quad \frac{1}{2} < x \le 1 \\ -(4x-5) & ; \quad 1 < x < \frac{5}{4} \\ 4x-5 & ; \quad \frac{5}{4} \le x < 2 \\ 6 & ; \quad x = 2 \end{cases}$$



function dis at 0, $0, \frac{1}{2}, 1, 2$

30 If f(x) = x + {-x} + [x] , where [x] is the integral part & {x} is the fractional part of x. Discuss the continuity of f in [– 2, 2].

Sol. $f(x) = x + \{-x\} + [x]$

$\because \{x\} = x - [x]$

$\{-x\} = -x - [-x]$

$f(x) = x + (-x - [-x] + [x])$

$f(x) = [x] - [-x]$ <
- $x - (-x) = 2x;\ x \in I$
- $[x] - (-[x] - 1) = 1 - 2[x];\ x \notin I$

$$f(x) = \begin{cases} 2x & ; \quad x \in I \\ 1 - 2[x] & ; \quad x \notin I \end{cases}$$

$$f(x) = \begin{cases} -4 & ; \quad x = -2 \\ 5 & ; \quad -2 < x < -1 \\ -2 & ; \quad x = -1 \\ 3 & ; \quad -1 < x < 0 \\ 0 & ; \quad x = 0 \\ 1 & ; \quad 0 < x < 1 \\ 2 & ; \quad x = 1 \\ -1 & ; \quad 1 < x < 2 \\ 4 & ; \quad x = 2 \end{cases}$$

so the function is discontinuous at all integers in [–2, 2].

31 Find the locus of (a, b) for which the function $f(x) = \begin{cases} ax - b & \text{for} \quad x \le 1 \\ 3x & \text{for} \quad 1 < x < 2 \\ bx^2 - a & \text{for} \quad x \ge 2 \end{cases}$ is continuous at x = 1 but discontinuous at x = 2.

Sol. conti at x = 1

$a - b = 3$ ...(1)

dis at x = 2

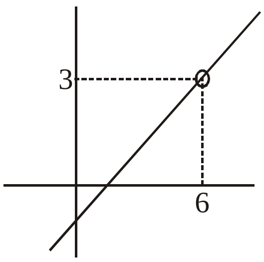
$\neq 4b - a$

$6 \neq 4b - 3 - b$

$6 \neq 3b - 3$

$\boxed{b \neq 3}$

$\boxed{a \neq 6}$



$(a, b) \neq (6, 3)$

$(x, y) \neq (6, 3)$ **Ans**

32 $f(x) = \dfrac{a^{\sin x} - a^{\tan x}}{\tan x - \sin x}$ for x > 0

$= \dfrac{ln(1 + x + x^2) + ln(1 - x + x^2)}{\sec x - \cos x}$ for x < 0, if f is continuous at x = 0, find 'a'

now if $g(x) = ln\left(2 - \dfrac{x}{a}\right) \cdot \cot(x - a)$ for x ≠ a, a ≠ 0, a > 0. If g is continuous at x = a then show that $g(e^{-1}) = -e$.

Sol. Since the function is conti at x = 0 then

$V.F.|_{x=0} = RHL|_{x=0} = LHL|_{x=0}$

$RHL|_{x=0} = \lim\limits_{x \to 0^+} f(x)$

$= \lim\limits_{x \to 0^+} \dfrac{a^{\sin x} - a^{\tan x}}{\tan x - \sin x}$

since the function is conti then

$f(0) = LHL|_{x=0} = RHL|_{x=0}$

$-\ell na = 1$

$$= \lim_{x\to 0^+} \frac{a^{\tan x}(a^{\sin x - \tan x} - 1)}{-1(\sin x - \tan x)}$$

$$\boxed{RHL|_{x=0} = -\ell na}$$

$$LHL|_{x=0} = \lim_{x\to 0^-} f(x)$$

$$= \lim_{x\to 0^-} \frac{\ell n(1+x+x^2) + \ell n(1-x+x^2)}{\sec x - \cos x}$$

$$= \lim_{x\to 0^-} \frac{\ell n((1+x+x^2)\,(1-x+x^2)).\cos x}{1-\cos 2x}$$

put $x = 0 - h$

$$= \lim_{h\to 0} \frac{\ell n(1+h^2+h^4)\cosh}{\sin^2 h}$$

$$= \lim_{h\to 0} (h^2+h^4)\frac{\cosh}{\sin^2 h}$$

$$= \lim_{h\to 0} \left(\frac{h}{\sinh}\right)^2 (1+h^2)\cosh$$

$$\boxed{a = \frac{1}{e}}$$

since $g(x)$ conti at $x = a$

$$g(a) = \lim_{x\to a} g(x)$$

$$= \lim_{x\to a} \ell n\left(2 - \frac{x}{a}\right)\cot(x-a)$$

$$= \lim_{x\to a} \frac{\ell n\left(2 - \frac{x}{a}\right)}{\tan(x-a)}$$

put $x = a + h$

$$= \lim_{h\to 0} \frac{\ell n\left(1 - \frac{h}{a}\right)}{\left(-\frac{h}{a}\right)} . \frac{h}{\tan}\left(-\frac{1}{a}\right)$$

$$g(a) = -\frac{1}{a}$$

$$\boxed{g\left(\frac{1}{e}\right) = -e}$$

33 Find the value of $\underset{x\to 0^+}{\text{Lim}}\ x^{\left(x^x - 1\right)}$. [Ans. 1]

[Sol. $l = \underset{x\to 0^+}{\text{Lim}}\ x^{(x^x - 1)}$ (0^0 form)

$$\ln l = \underset{x\to 0}{\text{Lim}}\ (x^x - 1).\ln x \ = \underset{x\to 0}{\text{Limit}} \frac{\left(e^{x\ln x} - 1\right)}{x \ln x}\ \underset{x\to 0}{\text{Limit}}\ x\ln x\ .\ln x$$

$$= \underset{x\to 0}{\text{Limit}}\ x(\ln x)^2 \qquad (\text{ as } x \to 0\ x \ln x \to 0)$$

$$= \underset{x\to 0}{\text{Limit}} \frac{(\ln x)^2}{1/x} = \underset{x\to 0}{\text{Limit}} -\frac{2\ln x}{x}.x^2 \qquad \text{(use Lopital's rule)}$$

$$= \underset{x\to 0}{\text{Limit}} -2\ln x . x = 0 \qquad \Rightarrow \qquad l = e^0 = 1$$

34 . . . . . . (x) = $\sum_{r=1}^{n} \tan\left(\frac{x}{2^r}\right) \sec\left(\frac{x}{2^{r-1}}\right)$; r , n ∈ N

$$g(x) = \underset{n\to\infty}{\text{Limit}} \frac{\ell n\left(f(x) + \tan\frac{x}{2^n}\right) - \left(f(x) + \tan\frac{x}{2^n}\right)^n \cdot \left[\sin\left(\tan\frac{x}{2}\right)\right]}{1 + \left(f(x) + \tan\frac{x}{2^n}\right)^n}$$

$= k$ for $x = \frac{\pi}{4}$ and the domain of g(x) is $(0, \pi/2)$.

where [ ] denotes the greatest integer function.

Find the value of k, if possible, so that g(x) is continuous at x = π/4. Also state the points of discontinuity of g (x) in (0 , π/4) , if any.

Sol. $$\tan\frac{x}{2}\sec x = \frac{\sin x/2}{\cos\frac{x}{2}.\cos x} = \frac{\sin\left(x - \frac{x}{2}\right)}{\cos\frac{x}{2}.\cos x} = \frac{\sin x\cos\frac{x}{2} - \cos x\sin\frac{x}{2}}{\cos\frac{x}{2}.\cos x} = \tan x - \tan\frac{x}{2}$$

$$\tan\frac{x}{2}\sec x = \tan x - \tan\frac{x}{2}$$

$$\tan\frac{x}{2^2}.\sec\frac{x}{2} = \tan\frac{x}{2} - \tan\frac{x}{2^2}$$

$$\tan\frac{x}{2^3}.\sec\frac{x}{2^2} = \tan\frac{x}{2^2} - \tan\frac{x}{2^3}$$

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$$\tan\frac{x}{2^n}.\sec\frac{x}{2^{n-1}} = \tan\frac{x}{2^{n-1}} - \tan\frac{x}{2^n}$$

$$f(x) = \tan x - \tan\left(\frac{x}{2^n}\right)$$

$$f(x) + \tan\left(\frac{x}{2^n}\right) = \tan x \qquad ...(1)$$

using (1)

$$g(x) = \begin{cases} \lim\limits_{n\to\infty} \dfrac{\ell n(\tan x) - (\tan x)^n\left[\sin\left(\tan\frac{x}{2}\right)\right]}{1 + (\tan x)^n} & ; \quad x \neq \dfrac{\pi}{4} \\ k & ; \quad x = \dfrac{\pi}{4} \end{cases}$$

$$g(x) = \begin{cases} \lim\limits_{h\to\infty} \dfrac{\ell n(\tan x)}{1+(\tan x)n} & ; \quad x \neq \dfrac{\pi}{4} \\ k & ; \quad x = \dfrac{\pi}{4} \end{cases}$$

k = 0

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & ; \quad x < 1 \\ 1 & ; \quad x = 1 \\ \infty & ; \quad x > 1 \end{cases}$$

$$\lim_{n\to\infty} (\tan x)^n = \begin{cases} 0 & ; \quad x < \dfrac{\pi}{4} \\ 1 & ; \quad x = \dfrac{\pi}{4} \\ \infty & ; \quad x > \dfrac{\pi}{4} \end{cases}$$

35 Let f be continuous on the interval [0, 1] to R such that f (0) = f (1). Prove that there exists a point c in $\left[0, \frac{1}{2}\right]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$

Sol. Consider a conti function

$g(x) = f\left(x + \frac{1}{2}\right) - f(x)$; g is conti $\forall\, x \in \left[0, \frac{1}{2}\right]$

Now

$$g(0) = f\left(\frac{1}{2}\right) - f(0) \Rightarrow g(0) = f\left(\frac{1}{2}\right) - f(1)$$

$$g\left(\frac{1}{2}\right) g(1) - f\left(\frac{1}{2}\right) \Rightarrow g(0) = f(1) - f\left(\frac{1}{2}\right)$$

since g is continuous and g(0) and $g\left(\frac{1}{2}\right)$ are of opposite sign hence the equation g(x) = 0 must have at least one root in $\left[0, \frac{1}{2}\right]$.

$\therefore$ for some $c \in \left[0, \frac{1}{2}\right]; g(c) = 0$

$$\Rightarrow f\left(c+\frac{1}{2}\right) = f(c)$$

36 Consider the function $g(x) = \begin{cases} \dfrac{1-a^x + xa^x \ell na}{a^x x^2} & ; \ x < 0 \\ \dfrac{2^x a^x - x\ell n\,2 - x\ell n\,a - 1}{x^2} & ; \ x > 0 \end{cases}$

where a > 0, find the value of 'a' & 'g(0)' so that the function g(x) is continuous at x = 0.

Sol. $LHL|_{x=0} = \lim\limits_{x\to 0^-} g(x)$

$$= \lim_{x\to 0^-}\left(\frac{1-a^x + xa^x\ell na}{a^x a^2}\right)$$

put x = a – h

$$= \lim_{x\to 0}\left(\frac{1-a^{-h} - ha^{-h}\ell na}{a^{-h}h^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{a^h - 1 - h\ell na}{2h}\right); \frac{0}{0} \text{form}$$

$$= \lim_{h\to 0}\left(\frac{a^h\ell na - 0 - \ell na}{2h}\right); \frac{0}{0} \textbf{ Ans}$$

$$= \lim_{h\to 0}\left(\frac{a^h(\ell na)^2}{2}\right)$$

$$\boxed{LHL\,|_{x=0} = \frac{(\ell na)^2}{2}}$$

$RHL|_{x=0} = \lim\limits_{x\to 0^+} g(x)$

$$= \lim_{x\to 0^+}\left(\frac{2^x a^x - x\ell n2 - x\ell na - 1}{x^2}\right)$$

put x = 0 + h

$$= \lim_{h\to 0}\left(\frac{(2a)^h - h\ell n2 - h\ell na - 1}{h^2}\right); \frac{0}{0} \text{form}$$

$$\boxed{RHL|_{x=0} = \frac{(\ell n2a)^2}{2}}$$

since the function is conti

$g(0) = LHL|_{x=0} = RHL|_{x=0}$

$$\frac{(\ell n(2a))^2}{2} = \frac{(\ell na)^2}{2}$$

$(\ell n\ 2a + \ell na)\ (\ell n\ 2a - \ell na) = 0$

$\ell n(2a^2).\ \ell n2 = 0$

$\ell n2a^2 = 0$

$2a^2 = 1,\ a = \pm\dfrac{1}{\sqrt{2}}\ ;\ a > 0$

$$\boxed{a = \frac{1}{\sqrt{2}}}$$

$$\therefore\ g(0) = \frac{(\ell n2a)^2}{2}$$

$$= \frac{1}{2}\left(\ell n2.\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{1}{2}(\ell n\sqrt{2})^2$$

$= \lim_{h\to 0}\frac{(2a)^h \ell n2a - \ell na2}{2h}; \frac{0}{0}$ form $\qquad = \frac{1}{2}\left(\frac{1}{4}(\ell n2)^2\right)$

$= \lim_{h\to 0}\frac{(2a)^h(\ell n2a)^2}{2} \qquad = \frac{1}{8}(\ell n2)^2$

37 A function $f: R \to R$ satisfies the equation $f(x+y) = f(x).\ f(y)$ for all x,y in R and $f(x) \neq 0$ for any x in R. Let the function the differentiable at $x = 0$ and $f'(0) = 2$. Show that $f'(x) = 2f(x)$ for all x in R. Hence determine $f(x)$.

Sol Given that $f(x+y) = f(x).f(y)$ for all $x \in R$ ...(1)

Putting $x = y = 0$ in (1), we get

$f(0)\{f(0)-1\} = 0 \qquad \Rightarrow \qquad f(0) = 0$ or $f(0) = 1$

If $f(0) = 0$, then $f(x) = f(x+0) = f(x).f(0) = 0$ for all $x \in R$

Which is not true (given $f(x) \neq 0$)

So, $f(0) = 1$

$$\therefore \quad f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h}$$

$$= f(x)\lim_{h\to 0}\frac{f(h)-1}{h}$$

$$= f(x)\lim_{h\to 0}\frac{f(x)-f(0)}{h-0} \qquad (\because\ f(0)=1)$$

$$= f(x)f'(0) = 2f(x) \qquad (\because\ f'(0)=2)$$

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = 2$$

Integrating both sides w.r.t.x and taking limit 0 to x

$$\int_0^x \frac{f'(x)}{f(x)}dx = \int_0^x 2\ dx$$

$\Rightarrow \quad \ln f(x) - \ln f(0) = 2x \qquad \Rightarrow \quad \ln f(x) - \ln 1 = 2x$

$\Rightarrow \quad \ln f(x) - 0 = 2x \qquad \therefore \quad f(x) = e^{2x}.$

38 Let f be a function such that $f(x+f(y)) = f(f(x)) + f(y) \quad \forall\ x,y \in R$ and $f(h) = h$ for $0 < h < \varepsilon$ where $\varepsilon > 0$, then determine $f'(x)$ and $f(x)$.

Sol Given $f(x+f(y)) = f(f(x)+f(y))$ ....(1)

Putting $x = y = 0$ in (1), then

$f\left(0+f(0)\right)=f\left(f(0)\right)+f(0) \qquad \Rightarrow \qquad f\left(f(0)\right)=f\left(f(0)\right)+f(0)$

$\therefore \qquad f(0)=0$ ....(2)

Now $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ $(\text{for } 0<h<\varepsilon)$

$=\lim_{h\to 0}\frac{f(h+x)-f(x)}{h}$

$=\lim_{h\to 0}\frac{f\left(f(h)\right)}{h}$ $\left(\text{form } (1)\right)$

$=\lim_{h\to 0}\frac{f(h)}{h}$ $\left(\because f(h)=h\right)$

$=\lim_{h\to 0}\frac{h}{h}=1$ $\left(\because f(h)=h\right)$

Integrating both sides with limites 0 to x then $f(x)=x$

$\therefore \qquad f'(x)=1.$

39 Let $f(x)=\begin{cases}-2 & , \ -3\le x\le 0\\ x-2 & , \ 0<x\le 3\end{cases}$, where $g(x)=f(|x|)+|f(x)|$. Test the differentiability of $g(x)$ in the interval $(-3,3)$.

Sol From the given function

$f(|x|)=\begin{cases}-x-2 & \text{for} & -3\le x\le 0\\ x-2 & \text{for} & 0<x\le 3\end{cases}$ and $|f(x)|=\begin{cases}2 & \text{for} & -3\le x\le 0\\ -x+2 & \text{for} & 0<x\le 2\\ x-2 & \text{for} & 2<x\le 3\end{cases}$

$\therefore \qquad g(x)=f(|x|)+|f(x)|$

$=\begin{cases}-x & \text{for} & -3\le x\le 0\\ 0 & \text{for} & 0<x\le 2\\ 2x-4 & \text{for} & 2<x\le 3\end{cases}$

Check the differentiability

At

$x=0:$ $\qquad Lg'(0)=\lim_{h\to 0}\frac{g(0-h)-g(0)}{-h}$

$=\lim_{h\to 0}\frac{-(0-h)-0}{-h}=-1$

$Rg'(0)=\lim_{h\to 0}\frac{g(0+h)-g(0)}{h}$

$=\lim_{h\to 0}\frac{(0-0)}{h}=0$

$\therefore \quad Lg'(0) \neq Rg'(0)$

$\therefore \quad g(x)$ is not differentiable at $x = 0$

Check at

$x = 2$: $\quad Lg'(2) = \lim_{h \to 0} \frac{g(2-h) - g(2)}{-h}$

$$= \lim_{h \to 0} \frac{0-0}{-h} = 0$$

and $\quad Rg'(2) = \lim_{h \to 0} \frac{g(2+h) - g(2)}{h}$

$$= \lim_{h \to 0} \frac{2(2+h) - 4 - 0}{h} = 2$$

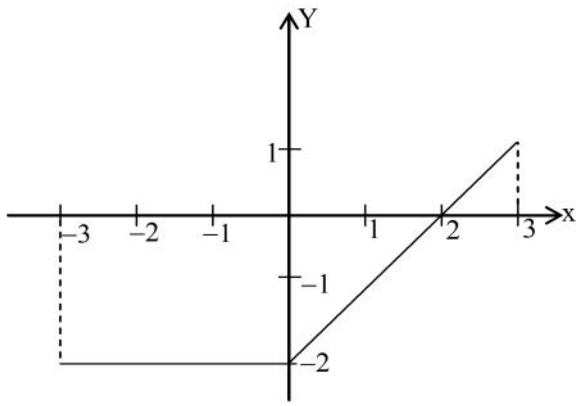
$\therefore \quad Lg'(2) \neq Rg'(2)$

Hence $g(x)$ is not differentiable at $x = 2$.

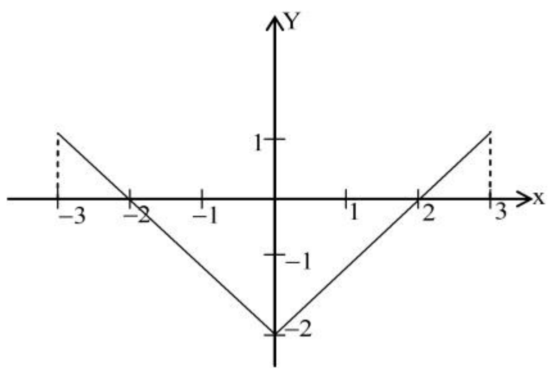
Graphical method :

$$\because \quad f(x) = \begin{cases} -2 & ; \quad -3 \le x \le 0 \\ x - 2 & ; \quad 0 < x \le 3 \end{cases}$$

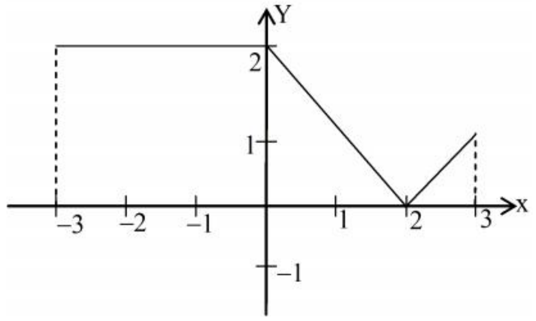
Graph of f(x) :



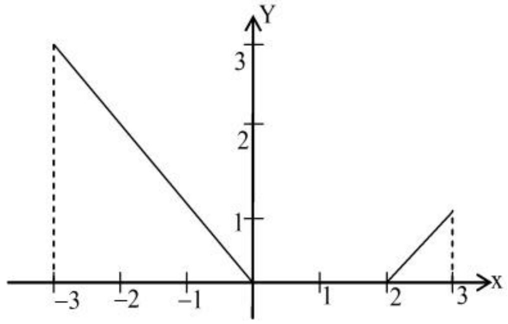
Graph of $f(|x|)$:



Graph of $|f(x)|$:



Graph of $g(x)=|f(x)|+f(|x|)$:



It is clear from the graph that g(x) is not differentiable at $x=0$ and 2.

40 Let $f:R\to R$ is a real valued function $\forall\ x,y\in R$ such that $|f(x)-f(y)|\le|x-y|^3$.

Prove that $h(x)=\int f(x)dx$ is continuous function of $x\ \ \forall\ \ x\in R$.

Sol Since $|f(x)-f(y)|\le|x-y|^3 \qquad x\ne y$

$\therefore \quad \left|\frac{f(x)-f(y)}{x-y}\right|\le|x-y|^2$

Taking lim as $y\to x,$ we get

$$\lim_{y\to x}\left|\frac{f(x)-f(y)}{x-y}\right|\le\lim_{y\to x}|x-y|^2$$

$$\Rightarrow \quad \left|\lim_{y\to x}\frac{f(x)-f(y)}{x-y}\right|\le\left|\lim_{y\to x}(x-y)^2\right|$$

$\Rightarrow \quad |f'(x)|\le 0 \qquad \Rightarrow \quad |f'(x)|=0 \qquad (\because |f'(x)|\ge 0)$

$\therefore \quad f'(x)=0 \qquad \Rightarrow \quad f(x)=c\ (\text{constant})$

$\therefore \quad h(x)=\int f(x)dx=\int c\ dx=cx+d$ where d is constant of integration.

$\therefore \quad h(x)$ is a linear function of x which is continuous for all $x\in R$.

41 Let $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$ for all real x and y. If $f'(0)$ exists and equals -1 and $f(0)=1,$

then find $f(2)$.

Sol Since $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$ ....(1)

$$\therefore \quad f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f\left(\frac{2x+2h}{2}\right)-f\left(\frac{2x+0}{2}\right)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{f(2x)+f(2h)}{2}-\frac{f(2x)+f(0)}{2}}{h} \quad \text{[from (1)]}$$

$$=\lim_{h\to 0}\frac{f(2h)-f(0)}{2h-0}$$

$$=f'(0)$$

$$=-1 \quad \forall \; x\in R \quad \text{(given)}$$

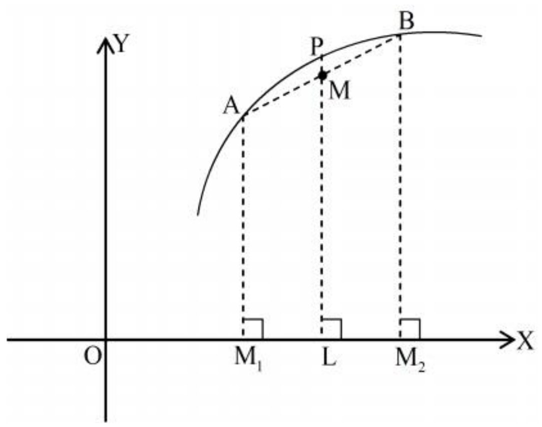
Integrating, we get $f(x)=-x+c$

Putting $x=0$, then $f(0)=0+c=1$ (given)

$\therefore \quad c=1$ then $f(x)=1-x$ $\quad \therefore \quad f(2)=1-2=-1$

Graphical method :

Suppose $A(x,f(x))$ and $B(y,f(y))$ be any two points on the curve $y=f(x)$.



If M is the mid-point of AB then co-ordinates of M are $\left(\frac{x+y}{2},\frac{f(x)+f(y)}{2}\right)$

According to the graph, co-ordinates of P are $\left(\frac{x+y}{2},f\left(\frac{x+y}{2}\right)\right)$ and $PL>ML$

$$\Rightarrow \quad f\left(\frac{x+y}{2}\right)>\frac{f(x)+f(y)}{2}$$

But given $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$ which is possible when $P\to M$

i.e. P lies on AB. Hence $y = f(x)$ must be a linear function.

Let $f(x) = ax + b$ $\Rightarrow$ $f(0) = 0 + b = 1$ (given)

and $f'(x) = a$ $\Rightarrow$ $f'(0) = a = -1$ (given)

$\therefore$ $f(x) = -x + 1$ $\therefore$ $f(2) = -2 + 1 = -1$.

42 Let $f\left(\frac{x+y}{n}\right) = \frac{f(x)+f(y)}{n}$ $\forall$ $x, y \in R; n \neq 0, 2$ and if $f'(0) = k$ (A finite quantity) then prove that $f(x) = kx$ $\forall$ $x \in R$.

Sol Given $f\left(\frac{x+y}{n}\right) = \frac{f(x)+f(y)}{n}$ ...(1)

Putting $x = y = 0$, we get $(n-2)f(0) = 0$

$\therefore$ $f(0) = 0$ $(\because n - 2 \neq 0)$

$$\therefore \quad f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f\left(\frac{nx+nh}{n}\right)-f\left(\frac{nx+0}{n}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{f(nx)+f(nh)}{n} - \frac{f(nx)+f(0)}{n}}{h} \quad \text{[from (1)]}$$

$$= \lim_{h\to 0}\frac{f(nh)-f(0)}{nh-0}$$

$\Rightarrow$ $f'(x) = k$

On integrating we get $f(x) = kx + c$

Putting $x = 0$, then $f(0) = 0 + c = 0$ $(\because f(0) = 0)$

$\therefore$ $c = 0$ then $f(x) = kx$.

43 If $f\left(\frac{x+y}{3}\right) = \frac{2+f(x)+f(y)}{3}$ for all real x and y and $f'(2) = 2$ then determine $y = f(x)$.

Sol $\because$ $f\left(\frac{x+y}{3}\right) = \frac{2+f(x)+f(y)}{3}$ ...(1)

Differentiating both sides w.r.t. x treating y as constant,

then $f'\left(\frac{x+y}{3}\right)\left(\frac{1}{3}\right) = \frac{2+f'(x)+0}{3}$

Now replacing x by 0 and y by 3x, then

$f'(x) = f'(0) = c$ (say)

At $x = 2$, $f'(2) = c = 2$ (given)

$\therefore$ $f'(x) = 2$

On integrating we get $f(x) = 2x + d$

Putting $x = 0,$ then $f(0) = 0 + d = 2$ [from (1)]

$\therefore \quad f(x) = 2x + 2$

Hence $y = 2x + 2.$

44 If $f\left(\frac{x+2y}{3}\right) = \frac{f(x)+2f(y)}{3} \quad \forall \; x, y \in R$ and $f'(0) = 1$; prove that $f(x)$ is continuous for all $x \in R.$

Sol $\because \quad f\left(\frac{x+2y}{3}\right) = \frac{f(x)+2f(y)}{3}$

Differentiating both sides w.r.t. x treating y as constant

$$f'\left(\frac{x+2y}{3}\right) \cdot \frac{1}{3} = \frac{f'(x)+0}{3}$$

and replacing x by 0 and y by $\frac{3x}{2}$

then $f'(x) = f'(0) = 1$ (given)

On integrating, we get

$f(x) = x + d, d$ is constant of integration which is linear function in x and hence it is always continuous function for all x.

45 If $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$ for all $x, y \in R$ and $xy \neq 1$ and $\lim_{x\to 0} \frac{f(x)}{x} = 2,$ find $f\left(\sqrt{3}\right)$ and $f'(-2).$

Sol Given $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$

Putting $x = 0, y = 0,$ we get $f(0) = 0$ ...(1)

And putting $y = -x,$ we get $f(x) + f(-x) = f(0) = 0$

$\therefore \quad f(x) = -f(-x)$ ...(2)

Now $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x+h) + f(-x)}{h}$$

$$= \lim_{h\to 0} \frac{f\left(\frac{h}{1+x(x+h)}\right)}{\frac{h}{1+x(x+h)}} \cdot \frac{1}{1+x(x+h)}$$

$$= 2 \cdot \frac{1}{1+x^2} \qquad \left(\because \lim_{x \to 0} \frac{f(x)}{x} = 2\right)$$

$$= \frac{2}{1+x^2}$$

$\therefore \quad f(x) = 2\tan^{-1} x + c$ or $f(0) = 2\tan^{-1} 0 + c = 0$

$\Rightarrow \quad 0 = 0 + c$ $\therefore \quad c = 0$

then $f(x) = 2\tan^{-1} x$

$\therefore \quad f(\sqrt{3}) = 2\tan^{-1}(\sqrt{3}) = \frac{2\pi}{3}$ and $f'(-2) = \frac{2}{1+(-2)^2} = \frac{2}{5}$.

46 Let $f(x+y) = f(x) + f(y) + 2xy - 1$ for all $x, y \in R$. If $f(x)$ is differentiable and $f'(0) = \sin\phi$ then prove that $f(x) > 0 \ \forall \ x \in R$.

Sol Given $f(x+y) = f(x) + f(y) + 2xy - 1 \quad \forall \ x, y \in R$ ...(1)

Putting $x = y = 0$ in (1), we get

$$f(0) = 1 \qquad ...(2)$$

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x) + f(h) + 2xh - 1 - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(h) + 2xh - 1}{h}$$

$$= \lim_{h \to 0} \frac{f(h) - 1}{h} + \lim_{h \to 0} \left(\frac{2xh}{h}\right)$$

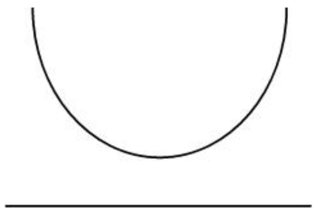
$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h} + \lim_{h \to 0} (2x)$$

$$= f'(0) + 2x$$

$$= \sin\phi + 2x \qquad (\because f(0) = \sin\phi)$$

Integrating both sides w.r.t. x and taking limit 0 to x, then

$$\int_0^x f'(x)\,dx = \int_0^x (\sin\phi + 2x)\,dx$$



$\Rightarrow \quad f(x) - f(0) = x\sin\phi + x^2$

$\Rightarrow \quad f(x) = x^2 + x\sin\phi + 1 \qquad (\because f(0) = 1)$

Here coefficient of x^2 is $1 > 0$ and Discriminant $D = \sin^2\phi - 4 < 0$.

Hence it is clear from graph $f(x) > 0 \quad \forall \ x \in R$.

47 Let f be a one-one function such that $f(x)f(y)+2=f(x)+f(y)+f(xy)\ \forall\ x,y\in R\sim\{0\}$ and $f(0)=1, f'(1)=2$ then prove that $3\int f(x)dx-x(f(x)+2)$ is constant.

Sol We have $f(x)f(y)+2=f(x)+f(y)+f(xy)$ ...(1)

Putting $x=1$ and $y=1$, we get

$$(f(1))^2+2=3f(1)$$

$\therefore$ $f(1)=1,2$ $\Rightarrow$ $f(1)=2$ ...(2)

$f(1)\neq 1$ ($\because f(0)=1$ and f is one-one function)

In (1), replacing y by $\frac{1}{x}$

$$\therefore\quad f(x)f\left(\frac{1}{x}\right)+2=f(x)+f\left(\frac{1}{x}\right)+f(1)$$

$$\Rightarrow\quad f(x)f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)\qquad (\because f(1)=2)$$

$$\therefore\quad f(x)=1\pm x^n\ (x\in N)$$

$$\Rightarrow\quad f'(x)=\pm nx^{n-1}\qquad\Rightarrow\qquad f'(1)=\pm n=2$$

Taking positive sign $\Rightarrow$ $n=2$ then $f(x)=1+x^2$

Now, $3\int f(x)dx-x(f(x)+2)$

$$=3\int(1+x^2)dx-x(1+x^2+2)$$

$$=3\left(x+\frac{x^3}{3}\right)+c-3x-x^3$$

$=c=$ constant.

48 If $e^{-xy}f(xy)=e^{-x}f(x)+e^{-y}f(y)\ \forall\ x,y\in R^+$, and $f'(1)=e$, determine $f(x)$.

Sol Given $e^{-xy}f(xy)=e^{-x}f(x)+e^{-y}f(y)$ ...(1)

Putting $x=y=1$ in (1) we get $f(1)=0$ ...(2)

Now, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right)-f(x\cdot 1)}{h}$$

$$=\lim_{h\to 0}\frac{e^{x+h}\cdot\left\{e^{-x}f(x)+e^{-1-\frac{h}{x}}f\left(1+\frac{h}{x}\right)\right\}-e^x\left(e^{-x}f(x)+e^{-1}f(1)\right)}{h}$$

$$= \lim_{h\to 0} \frac{e^h f(x) + e^{x+h-1-\frac{h}{x}} f\left(1+\frac{h}{x}\right) - f(x) - e^{x-1} f(1)}{h}$$

$$= f(x) \lim_{h\to 0}\left(\frac{e^h - 1}{h}\right) + e^{(x-1)} \lim_{h\to 0} \frac{e^{h-\frac{h}{x}} f\left(1+\frac{h}{x}\right)}{x \cdot \frac{h}{x}} \qquad (\because f(1) = 0)$$

$$= f(x).1 + e^{x-1} \cdot \frac{f'(1)}{x}$$

$$= f(x) + \frac{e^{x-1}.e}{x} \qquad (\because f'(1) = e)$$

$$f'(x) = f(x) + \frac{e^x}{x} \qquad \Rightarrow \qquad e^{-x} f'(x) - e^{-x} f(x) = \frac{1}{x}$$

$$\Rightarrow \qquad \frac{d}{dx}\left(e^{-x} f(x)\right) = \frac{1}{x}$$

On integrating we have $e^{-x} f(x) = \ln x + c$ at $x = 1, c = 0$

$\therefore \qquad f(x) = e^x \ln x.$

49 Let $f: R \to R$, such that $f'(0) = 1$

and $f(x+y) = f(x) + f(y) + e^{x+y}(x+y) - xe^x - ye^y + 2xy \ \forall x, y \in R$ then determine $f(x)$.

Sol Given $f(x+y) = f(x) + f(y) + e^{x+y}(x+y) - xe^x - ye^y + 2xy$ ...(1)

Putting $x = y = 0$, we get $f(0) = 0$ ...(2)

Now, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x) + f(h) + e^{x+h}(x+h) - xe^x - he^h + 2xh - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(h) + xe^x(e^h - 1) + he^{x+h} - he^h + 2xh}{h}$$

$$= \lim_{h\to 0} \left\{\frac{f(h)}{h} + xe^x \frac{(e^h - 1)}{h} + e^{x+h} - e^h + 2x\right\}$$

$$= f'(0) + xe^x.1 + e^x - 1 + 2x$$

$$= 1 + xe^x + e^x + 2x - 1$$

$$= xe^x + e^x + 2x$$

Integrating both sides w.r.t. x with limit 0 to x

$\therefore \qquad f(x) - f(0) = xe^x - e^x + e^x + x^2$

$$f(x) - 0 = xe^x + x^2$$

Hence $f(x) = x^2 + xe^x$

50 Let $f(xy) = x\,f(y) + y\,f(x)$ for all $x, y \in R_+$ and $f(x)$ be differentiable in $(0,\infty)$ then determine $f(x)$.

Sol Given $f(xy) = xf(y) + yf(x)$

Differentiating both sides w.r.t. x treating y as constant,

$$f'(xy).y = f(y) + yf'(x)$$

Putting $y = x$ and $x = 1$, then $f'(x).x = f(x) + xf'(1)$

$$\Rightarrow \quad \frac{xf'(x) - f(x)}{x^2} = \frac{f'(1)}{x} \qquad \Rightarrow \quad \frac{d}{dx}\left(\frac{f(x)}{x}\right) = \frac{f'(1)}{x}$$

Integrating both sides w.r.t. x taking limit 1 to x,

$$\frac{f(x)}{x} - \frac{f(1)}{1} = f'(1)\{\ln x - \ln 1\}$$

$$\Rightarrow \quad \frac{f(x)}{x} - 0 = f'(1)\ln x \qquad (\because f(1) = 0)$$

Hence, $f(x) = f'(1)(x \ln x)$.

51 Let $f(xy) = f(x)f(y)\ \forall\ x, y \in R$ and f is differentiable at $x = 1$ such that $f'(1) = 1$ also $f(1) \neq 0$ then show that f is differentiable for all $x \neq 0$. Hence, determine $f(x)$.

Sol Given $f(xy) = f(x)f(y)$

Putting $x = y = 1$ then we get $f(1) = 1$.

Differentiating both sides w.r.t. x treating y as constant,

$$f'(xy).y = f'(x)f(y)$$

Replacing y by x and x by 1, then

$$f'(x).x = f'(1)f(x)$$

$$\Rightarrow \quad f'(x) = \frac{f(x)f'(1)}{x} = \frac{f(x)}{x} \qquad (\because f'(1) = 1)$$

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = \frac{1}{x}$$

Integrating both sides w.r.t. x and taking limit 1 to x, then

$$\int_1^x \frac{f'(x)}{f(x)}dx = \int_1^x \frac{1}{x}dx$$

$$\Rightarrow \quad \ln f(x) - \ln f(1) = \ln x - \ln 1 \qquad (\because f(1) = 1)$$

$$\Rightarrow \quad \ln f(x) - 0 = \ln x - 0 \qquad \therefore \quad f(x) = x.$$

52 If $2f(x) = f(xy) + f\left(\frac{x}{y}\right)$ for all $x, y \in R^+, f(1) = 0$ and $f'(1) = 1$, then find $f(e)$ and $f'(2)$.

Sol Given $2f(x) = f(xy) + f\left(\frac{x}{y}\right)$ ...(1)

Replacing x by y and y by x in (1), then

$$2f(y) = f(xy) + f\left(\frac{y}{x}\right) \qquad ...(2)$$

Subtract (2) from (1), we get

$$2\{f(x) - f(y)\} = f\left(\frac{x}{y}\right) - f\left(\frac{y}{x}\right) \qquad ...(3)$$

Putting $x = 1$ in (1) then $2f(1) = f(y) + f\left(\frac{1}{y}\right) = 0 \qquad (\because\ f(1) = 0)$

$\therefore \quad f(y) = -f\left(\frac{1}{y}\right) \qquad \therefore \quad f\left(\frac{y}{x}\right) = -f\left(\frac{x}{y}\right)$ ...(4)

Now from (3) and (4), we get

$$2\{f(x) - f(y)\} = 2f\left(\frac{x}{y}\right)$$

or $$f(x) - f(y) = f\left(\frac{x}{y}\right) \qquad ...(5)$$

Now, $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f\left(1 + \frac{h}{x}\right)}{h} \qquad \text{[From (5)]}$$

$$= \lim_{h\to 0} \frac{f\left(1 + \frac{h}{x}\right)}{\frac{h}{x}\cdot x} = \frac{1}{x} f'(1) = \frac{1}{x} \qquad \{\because f'(1) = 1\}$$

$\therefore \quad f'(x) = \frac{1}{x} \qquad \Rightarrow \quad f'(2) = \frac{1}{2}$

and $f(x) = \ln x + \ln c$ for $x = 1$, and $f(1) = \ln 1 + \ln c$

$\Rightarrow \quad 0 = 0 + \ln c \qquad \therefore \quad \ln c = 0$

then $f(x) = \ln x \qquad \therefore \quad f(e) = \ln e = 1$.

$p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$. If $|p(x)| \le |e^{x-1} - 1|$ for all $x \ge 0$, prove that $|a_1 + 2a_2 + ... + na_n| \le 1$.

Sol Given $p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$

$\therefore \quad p'(x) = 0 + a_1 + 2a_2x + ... + na_nx^{n-1}$

$\Rightarrow \quad p'(1) = a_1 + 2a_2 + ... + na_n$ ...(1)

Now, $\left|p(1)\right| \le \left|e^{1-1}-1\right|$

$$= \left|e^0 - 1\right| = \left|1-1\right| = 0$$

$\Rightarrow \quad \left|p(1)\right| \le 0 \quad \Rightarrow \quad p(1) = 0 \qquad \left(\because \left|p(1)\right| \ge 0\right)$

As $\quad \left|p(x)\right| \le \left|e^{x-1}-1\right|$

we get $\left|p(1+h)\right| \le \left|e^h - 1\right| \;\; \forall\, h > -1, h \ne 0$

$\Rightarrow \quad \left|p(1+h) - p(1)\right| \le \left|e^h - 1\right| \qquad \left(\because p(1) = 0\right)$

$$\Rightarrow \quad \left|\frac{p(1+h)-p(1)}{h}\right| \le \left|\frac{e^h-1}{h}\right|$$

Taking limit as $h \to 0$, then

$$\Rightarrow \quad \lim_{h\to 0}\left|\frac{p(1+h)-p(1)}{h}\right| \le \lim_{h\to 0}\left|\frac{e^h-1}{h}\right|$$

$$\Rightarrow \quad \left|\lim_{h\to 0}\frac{p(1+h)-p(1)}{h}\right| \le \left|\lim_{h\to 0}\frac{e^h-1}{h}\right|$$

$$\Rightarrow \quad \left|p'(1)\right| \le 1$$

$\Rightarrow \quad \left|a_1 + 2a_2 + \ldots + na_n\right| \le 1 \qquad$ [from (1)]

54 Let $f\left(\frac{xy}{2}\right) = \frac{f(x)f(y)}{2}$ for all real x and y. If $f(1) = f'(1)$, show that $f(x) + f(1-x) =$ constant, for all non-zero real x.

Sol Given $f\left(\frac{xy}{2}\right) = \frac{f(x)f(y)}{2}$

Replacing x by 2x and y by 1, we get

$$2f(x) = f(2x)f(1) \qquad \ldots(1)$$

and,

$$f\left(\frac{x+y}{2}\right) = f\left(\frac{x\left(1+\frac{y}{x}\right)}{2}\right) = \frac{f(x)f\left(1+\frac{y}{x}\right)}{2}, x \ne 0 \qquad \ldots(2)$$

now,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(\frac{2x+2h}{2}\right) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(2x)f\left(1+\frac{h}{x}\right)}{2} - f(x)}{h}$$ [from (2)]

$$= \lim_{h\to 0} \frac{f(2x)f\left(1+\frac{h}{x}\right) - 2f(x)}{2h}$$

$$= \lim_{h\to 0} \frac{f(2x)f\left(1+\frac{h}{x}\right) - f(2x)f(1)}{2h}$$ [from (1)]

$$= \frac{f(2x)}{2} \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right) - f(1)}{x \cdot \frac{h}{x}}$$

$$= \frac{f(2x)}{2x} \cdot f'(1)$$

$$= \frac{2f(x)}{f(1).2x} \cdot f'(1) = \frac{f(x)}{x}$$ $(\because f'(1) = f(1))$

$$= \frac{f'(x)}{f(x)} = \frac{1}{x}$$

Integrating both sides w.r.t. x, we get

$$\ln f(x) = \ln x + \ln c$$

$\Rightarrow$ $f(x) = cx$ (c is constant > 0)

$\therefore$ $f(x) + f(1-x) = cx + c(1-x) = cx + c - cx = c = \text{constant}.$

55 Let $f(x) = x^3 - x^2 + x + 1$ and $g(x) = \max\{f(t) : 0 \le t \le x\}, 0 \le x \le 1 = 3 - x, 1 < x \le 2.$

Discuss the continuity and differentiability of the function $g(x)$ in the interval $(0,2)$.

Sol Given $f(x) = x^3 - x^2 + x + 1$

$\therefore$ $f'(x) = 3x^2 - 2x + 1$

$$= 3\left\{x^2 - \frac{2x}{3} + \frac{1}{3}\right\}$$

$$= 3\left\{\left(x - \frac{1}{3}\right)^2 + \frac{2}{9}\right\} > 0$$

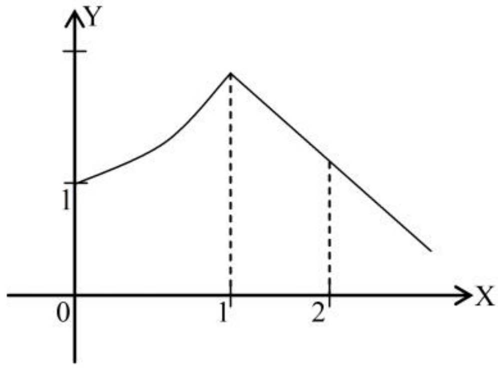
$\therefore$ $f(x)$ is strictly increasing in (0,2)

$\therefore$ maximum value of $f(t)$ in $0 \le t \le x$ is $f(x)$

$\therefore \qquad g(x) = \begin{cases} f(x) & , \quad 0 \le x \le 1 \\ 3 - x & , \quad 1 < x \le 2 \end{cases}$

$= \begin{cases} x^3 - x^2 + x + 1 & , \quad 0 \le x \le 1 \\ 3 - x & , \quad 1 < x \le 2 \end{cases}$

Graph of $g(x)$:



Clearly, $g(x)$ is continuous for all $x \in (0,2)$ and differentiable at all points in this interval except $x = 1$.

56 Let $f(x) = x^3 - 9x^2 + 15x + 6$, and $g(x) = \begin{cases} \min f(t) : 0 \le t \le x & , \quad 0 \le x \le 6 \\ x - 18 & , \quad x > 6 \end{cases}$, then draw the graph of $g(x)$ and discuss the continuity and differentiability of $g(x)$.

Sol $\because \qquad f(x) = x^3 - 9x^2 + 15x + 6,$

$\therefore \qquad f'(x) = 3x^2 - 18x + 15 = 3(x^2 - 6x + 5) = 3(x-1)(x-5)$

If $\qquad f'(x) > 0$ then $x \in (-\infty, 1) \cup (5, \infty)$

and if $f'(x) < 0$ then $x \in (1,5)$

Hence $f(x)$ is increasing in $(-\infty,1) \cup (5,\infty)$ and decreasing in (1,5).

Now, $f(x) = 6 \qquad \Rightarrow \qquad x^3 - 9x^2 + 15x + 6 = 6$

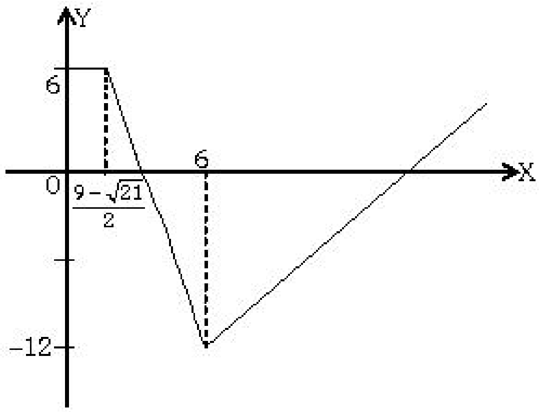
$\Rightarrow \qquad x^3 + 9x^2 + 15x = 0 \qquad \Rightarrow \qquad x(x^2 - 9x + 15) = 0$

$\Rightarrow \qquad x = 0, \dfrac{9 \pm \sqrt{21}}{2}$

$\Rightarrow \qquad x = 0, \dfrac{9 - \sqrt{21}}{2} \qquad \left(x \ne \dfrac{9 + \sqrt{21}}{2}, \because \dfrac{9 - \sqrt{21}}{2} > 6 \right)$

$$\therefore \qquad g(x)=\begin{cases} 6 & , \quad 0 \le x < \dfrac{9-\sqrt{21}}{2} \\ x^3-9x^2+15x+6 & , \quad \dfrac{9-\sqrt{21}}{2} \le x \le 6 \\ x-18 & , \quad x > 6 \end{cases}$$

Graph of $g(x)$:



Clearly g(x) is continuous in $[0,\infty)$ and differentiable at all points in this interval other than $\dfrac{9-\sqrt{21}}{2}$ and 6.

57 Let $f(x)=\begin{cases} b\sin^{-1}\left(\dfrac{x+c}{2}\right) & , \quad -\dfrac{1}{2} < x < 0 \\ \dfrac{1}{2} & , \quad x=0 \\ \dfrac{e^{ax/2}-1}{x} & , \quad 0 < x < \dfrac{1}{2} \end{cases}$, If $f(x)$ is differentiable at $x=0$. Find the value of a also prove that $64b^2=4-c^2$.

Sol $Rf'(0)=\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\dfrac{\dfrac{e^{\frac{ah}{2}}-1}{h}-\dfrac{1}{2}}{h}$

$$=\lim_{h\to 0}\frac{\dfrac{a}{2}\cdot\left(\dfrac{e^{\frac{ah}{2}}-1}{\frac{ah}{2}}\right)-\dfrac{1}{2}}{h}$$

at $h\to 0$ numerator must be = 0, then $\dfrac{a}{2}.1-\dfrac{1}{2}=0$

$\therefore \quad a = 1$

$$\Rightarrow \quad Rf'(0) = \lim_{h\to 0}\frac{\frac{e^{\frac{h}{2}}-1}{h}-\frac{1}{2}}{h} = \lim_{h\to 0}\frac{2\left(e^{\frac{h}{2}}-1\right)-h}{2h^2} = P\ (\text{say}) \qquad \ldots(1)$$

$$\therefore \quad P = \lim_{h\to 0}\frac{2\left(e^{\frac{h}{2}}-1\right)-h}{2h^2}$$

Replacing h by $-h$ then $P = \lim_{h\to 0}\frac{2\left(e^{-\frac{h}{2}}-1\right)+h}{2h^2}$ ...(2)

Adding (1) and (2) then $2P = \lim_{h\to 0}\frac{e^{\frac{h}{2}}+e^{-\frac{h}{2}}-2}{h^2} = \lim_{h\to 0}\frac{e^h-2e^{\frac{h}{2}}+1}{h^2 e^{\frac{h}{2}}}$

$$= \lim_{h\to 0}\left(\frac{e^{\frac{h}{2}}-1}{\frac{h}{2}}\right)^2 \cdot \frac{1}{4e^{\frac{h}{2}}} = \frac{1}{4}$$

$\therefore \quad P = \frac{1}{8} \qquad \Rightarrow \qquad Rf'(0) = \frac{1}{8}$ ...(3)

$$Lf'(0) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{b\sin^{-1}\left(\frac{-h+c}{2}\right)-\frac{1}{2}}{-h}$$

Now, at $h \to 0$ numerator must $\text{be} = 0$

$\therefore \quad b\sin^{-1}\left(\frac{c}{2}\right)-\frac{1}{2} = 0$

then,

$$Lf'(0) = b\lim_{h\to 0}\frac{\sin^{-1}\left(\frac{c-h}{2}\right)-\sin^{-1}\left(\frac{c}{2}\right)}{-h}$$

$$= b\lim_{h\to 0}\frac{\sin^{-1}\left\{\left(\frac{c-h}{2}\right)\sqrt{\left(1-\frac{c^2}{4}\right)}-\frac{c}{2}\sqrt{\left(1-\left(\frac{c-h}{2}\right)^2\right)}\right\}}{-h}$$

$$= b\lim_{h\to 0}\frac{\sin^{-1}\left\{\left(\frac{c-h}{2}\right)\sqrt{\left(1-\frac{c^2}{4}\right)}-\frac{c}{2}\sqrt{\left(1-\left(\frac{c-h}{2}\right)^2\right)}\right\}}{\left(\frac{c-h}{2}\right)\sqrt{\left(1-\frac{c^2}{4}\right)}-\frac{c}{2}\sqrt{\left(1-\left(\frac{c-h}{2}\right)^2\right)}}$$

$$\cdot\frac{\left\{\left(\frac{c-h}{2}\right)\sqrt{\left(1-\frac{c^2}{4}\right)}-\frac{c}{2}\sqrt{1-\left(\frac{c-h}{2}\right)^2}\right\}}{-h}$$

$$=-b\lim_{h\to0}\frac{\left\{\left(\frac{c-h}{2}\right)\sqrt{\left(1-\frac{c^2}{4}\right)}-\frac{c}{2}\sqrt{1-\left(\frac{c-h}{2}\right)^2}\right\}\left\{\left(\frac{c-h}{2}\right)\sqrt{\left(1-\frac{c^2}{4}\right)}+\frac{c}{2}\sqrt{1-\left(\frac{c-h}{2}\right)^2}\right\}}{h\left\{\left(\frac{c-h}{2}\right)\sqrt{\left(1-\frac{c^2}{4}\right)}-\frac{c}{2}\sqrt{1-\left(\frac{c-h}{2}\right)^2}\right\}}$$

$$=-b\lim_{h\to0}\frac{\left(\frac{c-h}{2}\right)^2\left(1-\frac{c^2}{4}\right)-\frac{c^2}{4}\left(1-\left(\frac{c-h}{2}\right)^2\right)}{h\left\{\left(\frac{c-h}{2}\right)\sqrt{\left(1-\frac{c^2}{4}\right)}+\frac{c}{2}\sqrt{1-\left(\frac{c-h}{2}\right)^2}\right\}}$$

$$=-b\lim_{h\to0}\frac{(2c-h)(-h)}{4h\left\{\left(\frac{c-h}{2}\right)\sqrt{\left(1-\frac{c^2}{4}\right)}+\frac{c}{2}\sqrt{\left\{1-\left(\frac{c-h}{2}\right)^2\right\}}\right\}}$$

$$=\frac{2bc}{4\left\{c\sqrt{\left(1-\frac{c^2}{4}\right)}\right\}}=\frac{b}{2\sqrt{\left(1-\frac{c^2}{4}\right)}} \qquad \ldots(5)$$

From (3) and (5),

$$\frac{1}{8}=\frac{b}{2\sqrt{\left(1-\frac{c^2}{4}\right)}}$$

$\Rightarrow \quad 64b^2=4-c^2$

58 Let $\alpha\in R.$ Prove that a function $f:R\to R$ is differentiable at $x=\alpha$ if and only if there is a function $g:R\to R$ which is continuous at α and satisfies $f(x)-f(\alpha)=g(x)(x-\alpha)$ for all $\alpha\in R.$

Sol Let $f:R\to R$ be differentiable at $x=\alpha\in R$, then

$\lim_{x\to\alpha}\frac{f(x)-f(\alpha)}{(x-\alpha)}=f'(\alpha)$ exists and finite.

i.e. $\quad Lf'(\alpha)=Rf'(\alpha)=f'(\alpha)$

$$\Rightarrow \quad \lim_{x\to\alpha-}\frac{f(x)-f(\alpha)}{(x-\alpha)}=\lim_{x\to\alpha+}\frac{f(x)-f(\alpha)}{(x-\alpha)}=f'(\alpha)$$

$$\lim_{x\to\alpha-}g(x)=\lim_{x\to\alpha+}g(x)=f'(\alpha) \qquad \{\because f(x)-f(\alpha)=g(x)(x-\alpha)\} \qquad \ldots(1)$$

Again $f'(\alpha)=\lim_{x\to\alpha}\frac{f(x)-f(\alpha)}{(x-\alpha)}$

$$=\lim_{x\to\alpha} g(x)=g(\alpha)$$

From (1) and (2), we get $\lim_{x\to\alpha-} g(x)=\lim_{x\to\alpha+} g(x)=g(\alpha)$

L.H.L = R.H.L = V.F.

$\Rightarrow$ $g(x)$ is continuous function at $x=\alpha\in R$.

59 Let $g(x)=0$ if $-e\le x<1$

$$=\left\{1+\frac{1}{3}\sin\left(\ln x^{2\pi}\right)\right\} \text{ if } 1\le x\le e.$$

where {} denotes the fractional part function and

$$f(x)=x\,g(x) \text{ for } g(x)=1+\frac{1}{3}\sin\left(\ln x^{2\pi}\right)$$

$$=x\left(g(x)+1\right) \quad \text{otherwise}$$

Discuss the continuity and differentiability of $f(x)$ over its domain.

Sol Given $g(x)=\left\{1+\frac{1}{3}\sin\left(\ln x^{2\pi}\right)\right\}$ for $1\le x\le e$

$$=0 \text{ for } -e\le x<1$$

i.e., $g(x)=1+\frac{1}{3}\sin\left(\ln x^{2\pi}\right)-\left[1+\frac{1}{3}\sin\left(\ln x^{2\pi}\right)\right]$

$$=\frac{1}{3}\sin\left(\ln x^{2\pi}\right)-\left[\frac{1}{3}\sin\left(\ln x^{2\pi}\right)\right], 1\le x\le e$$

$$=0, -e\le x<1$$

where [.] denotes the greatest integer function.

consider : $1\le x\le e$

$\Rightarrow$ $(1)^{2\pi}\le x^{2\pi}\le e^{2\pi}$ $\Rightarrow$ $\ln(1)\le\ln\left(x^{2\pi}\right)\le\ln\left(e^{2\pi}\right)$

$\Rightarrow$ $0\le\ln\left(x^{2\pi}\right)\le 2\pi$

Case I : If $0\le\ln\left(x^{2\pi}\right)\le\pi$ i.e. $1\le x\le\sqrt{e}$ then $0\le\sin\left(\ln\left(x^{2\pi}\right)\right)\le 1$

$\Rightarrow$ $0\le\frac{1}{3}\sin\left(\ln\left(x^{2\pi}\right)\right)\le\frac{1}{3}$ $\therefore$ $\left[\frac{1}{3}\sin\left(\ln\left(x^{2\pi}\right)\right)\right]=0$

$\therefore$ $g(x)=\frac{1}{3}\sin\left(\ln x^{2\pi}\right)$ for $1\le x\le\sqrt{e}$

Case II : If $\pi<\ln\left(x^{2\pi}\right)<2\pi$ i.e., $\sqrt{e}<x<e$ then $-1\le\sin\left(\ln\left(x^{2\pi}\right)\right)<0$

$\Rightarrow$ $-\frac{1}{3}\le\frac{1}{3}\sin\left(\ln\left(x^{2\pi}\right)\right)<0$ $\therefore$ $\left[\frac{1}{3}\sin\left(\ln\left(x^{2\pi}\right)\right)\right]=-1$

$\therefore$ $g(x)=1+\frac{1}{3}\sin\left(\ln\left(x^{2\pi}\right)\right)$ for $\sqrt{e}<x<e$

Case III : If $\ln\left(x^{2\pi}\right)=2\pi$ $\Rightarrow$ $x=e$ $\Rightarrow$ $g\left(x\right)=\left\{1\right\}=0$

Combining all cases, we get

$$f\left(x\right)=x\left(1+\frac{1}{3}\sin\left(\ln\left(x^{2\pi}\right)\right)\right) \qquad \text{for} \sqrt{e}<x<e$$

$$=x\left(1+\frac{1}{3}\sin\left(\ln\left(x^{2\pi}\right)\right)\right) \qquad \text{for} 1\le x\le\sqrt{e}$$

$$=x\left(1+0\right) \qquad \text{for} -e\le x<1$$

$$=x\left(1+0\right) \qquad \text{for } x=e$$

$\Rightarrow$
$$f\left(x\right)=x\left(1+\frac{1}{3}\sin\left(\ln\left(x^{2\pi}\right)\right)\right) \qquad \text{for } 1\le x\le e$$

$$=x$$

$\therefore$ f is differentiable in $\left(-e,1\right)$ and $\left(1,e\right)$

Check the differentiable of $f\left(x\right)$ at $x=1$.

$$Lf'\left(1\right)=\lim_{h\to 0}\frac{f\left(1-h\right)-f\left(1\right)}{-h}$$

$$=\lim_{h\to 0}\frac{\left(1-h\right)-1}{-h}=1$$

and
$$Rf'\left(1\right)=\lim_{h\to 0}\frac{f\left(1+h\right)-f\left(1\right)}{h}$$

$$=\lim_{h\to 0}\frac{\left(1+h\right)\cdot\left(1+\frac{1}{3}\sin\left(\ln\left(1+h\right)^{2\pi}\right)\right)-1}{h}$$

$$=\lim_{h\to 0}\frac{h+\frac{\left(1+h\right)}{3}\sin\left(\ln\left(1+h\right)^{2\pi}\right)}{h}$$

$$=\lim_{h\to 0}\left(1+\frac{\left(1+h\right)}{3}\frac{\sin\left\{\ln\left(1+h\right)^{2\pi}\right\}}{h}\right)$$

$$=1+\lim_{h\to 0}\frac{\left(1+h\right)}{3}\lim_{h\to 0}\frac{\sin\left(\ln\left(1+h\right)^{2\pi}\right)}{h}$$

$$=1+\lim_{h\to 0}\frac{\left(1+h\right)}{3}\lim_{h\to 0}\frac{\sin\left\{2\pi\ln\left(1+h\right)\right\}}{2\pi\ln\left(1+h\right)}\cdot\frac{2\pi\ln\left(1+h\right)}{h}$$

$$=1+\left(\frac{1+0}{3}\right).1.2\pi.1$$

$$=1+\frac{2\pi}{3}\cdot$$

Thus f is not differentiable at $x=1$.

Hence f is continuous and differentiable for all $x \in$ domain of except not differentiable at $x = 1$.

60 Suppose that f and g are non-constant differentiable real valued functions on R. If for every $x, y \in R, f(x+y) = f(x)f(y) - g(x)g(y),\ g(x+y) = g(x)f(y) + f(x)g(y)$ and $f'(0) = 0$ then prove that $\{f(x)\}^2 + \{g(x)\}^2 = 1\ \ \forall\ \ x \in R.$

Sol We have $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x+0)}{h}$

$$= \lim_{h\to 0} \frac{\{f(x)f(h) - g(x)g(h)\} - \{f(x)f(0) - g(x)g(0)\}}{h}$$

$$= \lim_{h\to 0} \frac{f(x)(f(h) - f(0))}{(h-0)} - \lim_{h\to 0} \frac{g(x)(g(h) - g(0))}{(h-0)}$$

$$= f(x)f'(0) - g(x)g'(0)$$

$$= 0 - g(x)g'(0) \qquad (\because f'(0) = 0)$$

$\therefore$ $f'(x) = -g(x)g'(0)$ ....(1)

and $g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{g(x+h) - g(x+0)}{h}$

$$= \lim_{h\to 0} \frac{\{g(x)f(h) + f(x)g(h)\} - \{g(x)f(0) + f(x)g(0)\}}{h}$$

$$= g(x)\lim_{h\to 0} \frac{f(h) - f(0)}{h-0} + f(x)\lim_{h\to 0} \frac{g(h) - g(0)}{h-0}$$

$$= g(x)f'(0) + f(x)g'(0)$$

$$= 0 + f(x)g'(0) \qquad (\because f'(0) = 0)$$

$= f(x)g'(0)$ ....(2)

Multiplying (1) by f(x) and (2) by g(x) and adding we get

$$f(x)f'(x) + g(x)g'(x) = 0$$

or $2f(x)f'(x) + 2g(x)g'(x) = 0$ on integrating we get

$\{f(x)\}^2 + \{g(x)\}^2 = c$ .....(3)

Putting $x = 0, y = 0$ in the given equation then

$$f(0) = \{f(0)\}^2 - \{g(0)\}^2 \qquad \text{and} \qquad g(0) = 2f(0)g(0)$$

or $g(0)\{2f(0) - 1\} = 0$ or $g(0) = 0$ or $f(0) = \frac{1}{2}$

If $g(0) = 0$, then $f(0) = (f(0))^2 - 0$ or $f(0) = 1$

and for $f(0) = \frac{1}{2}, \frac{1}{2} = \left(\frac{1}{2}\right)^2 - (g(0))^2$

$\Rightarrow \quad (g(0))^2 = -\frac{1}{4} \qquad \text{(Impossible)}$

Hence $f(0)=1$ and $g(0)=0$ from (3), $\{f(0)\}^2 + \{g(0)\}^2 = c$

$\Rightarrow \quad 1+0=c \qquad \therefore \quad c=1$

Hence $\{f(x)\}^2 + \{g(x)\}^2 = 1$.

61 Let $f(x)$ be a real valued function not identically zero such that

$f(x+y^n) = f(x) + \{f(y)\}^n$; $\forall$ $x, y \in R$ (where n is odd natural number > 1) and $f'(0) \geq 0$.

Find out the values of $f'(10)$ and $f(5)$.

Sol Given that $f(x+y^n) = f(x) + (f(y))^n$

Putting $x = y = 0$ $\quad \Rightarrow \quad f(0) = 0$

$$\therefore \quad f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h) - 0}{h}$$

$$= \lim_{h\to 0} \frac{f(h)}{h} = \lambda \text{ (say)} \qquad \text{...(1)}$$

Also,

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f\left(0 + \left(h^{1/n}\right)^n\right) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(0) + \left\{f\left(h^{1/n}\right)\right\}^n - f(0)}{h}$$

$$= \lim_{h\to 0} \left\{\frac{f\left(h^{1/n}\right)}{h^{1/n}}\right\}^n$$

$$= \lambda^n \qquad \text{[from (1)]}$$

From (1) and (2), $\quad \lambda = \lambda^n$

$\therefore \quad \lambda = -1, 0, 1 \qquad (\because \text{n is odd and } \lambda \in R)$

$\because \quad f'(0) \geq 0 \qquad (\because \lambda \neq -1)$

$\therefore \quad f'(0) = 0, 1$

Again $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{f\left(x + \left(h^{1/n}\right)^n\right) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x) + \left(f\left(h^{1/n}\right)\right)^n - f(x)}{h}$$

$$= \lim_{h\to 0} \left(\frac{f\left(h^{1/n}\right)}{h^{1/n}}\right)^n \qquad = \lambda^n$$

For $\lambda = 0, f'(x) = 0$

On intergrating we get $f(x) = c$

At $\quad x = 0, f(0) = c = 0 \qquad (\because f(0) = 0)$

$\therefore \quad f(x) = 0$

which is impossible as f(x) is not identically zero, i.e., $f(x) \neq 0$

and for $\lambda = 1 \qquad f'(x) = 1$

On intergrating w.r.t. x and taking limit 0 to x,

then $\int_0^x f'(x)dx = \int_0^x 1.dx$

$\Rightarrow \quad f(x) - f(0) = x \qquad \Rightarrow \quad f(x) - (0) = x \qquad (\because f(0) = 0)$

Hence $f(x) = x$ and $f'(x) = 1 \qquad \therefore \quad f'(10) = 1$ and $f(5) = 5.$

62 Let $a_1 > a_2 > a_3$ .... $a_n > 1$; $p_1 > p_2 > p_3$......... $> p_n > 0$; such that $p_1 + p_2 + p_3 + ... + p_n = 1$

Also $F(x) = \left(p_1 a_1^x + p_2 a_2^x + ... + p_n a_n^x\right)^{1/x}$. Compute

(a) $\lim_{x\to 0^+} F(x)$ **(b)** $\lim_{x\to\infty} F(x)$ **(c)** $\lim_{x\to -\infty} F(x)$ [**Ans. (a)** $a_1^{p_1} \cdot a_2^{p_2} .....a_n^{p_n}$; **(b)** a_1 ; **(c)** a_n]

[Sol.

(1) $\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} \left(p_1 a_1^x + p_2 a_2^x + ... + p_n a_n^x\right)^{1/x}$ $\quad (1^\infty$ form)

$_1 = e^l$ where $l = \lim_{x\to 0} \frac{p_1 a_1^x + p_2 a_2^x + ... + p_n a_n^x - 1}{x} \qquad \left(\frac{0}{0}\right)$

using L'Hospital's Rule

$l = \lim_{x\to 0} \left(p_1 \, ln\, a_1 \, a_1^x + p_2 \, ln\, a_2 \, a_2^x + ... + p_n \, ln\, a_n \, a_n^x\right)$

$= p_1 \, ln\, a_1 + p_2 \, ln\, a_2 + ... + p_n \, ln\, a_n$

$= ln\left(a_1^{p_1} \cdot a_2^{p_2} .....a_n^{p_n}\right)$

$\therefore \quad L_1 = e^l = a_1^{p_1} \cdot a_2^{p_2} .....a_n^{p_n}$ **Ans.**

(2) $\lim_{x\to\infty} F(x) = L_2 = \lim_{x\to\infty} \left(p_1 a_1^x + p_2 a_2^x + ... + p_n a_n^x\right)^{1/x}$ $\quad (\infty^0$ form) [only when $a_1 \, a_2$ etc. > 1]

$\therefore \quad ln\, L_2 = \lim_{x\to\infty} \frac{ln\left(p_1 a_1^x + p_2 a_2^x + ... + p_n a_n^x\right)}{x}$

using L'Hospital's Rule

$$_2 = \lim_{x\to\infty} \frac{\left(p_1\, ln\, a_1\, a_1^x + p_2\, ln\, a_2\, a_2^x + ... + p_n\, ln\, a_n\, a_n^x\right)}{p_1 a_1^x + p_2 a_2^x + ... + p_n a_n^x} \qquad ....(1)$$

dividing by a_1^x and taking limit, we get

$$\lim_{x\to\infty},\ \left(\frac{a_2}{a_1}\right)^x,\ \left(\frac{a_3}{a_2}\right)^x,\ \text{etc all vanishes as } x \to \infty$$

$$= \frac{p_1\, ln\, a_1}{p_1} = ln\, a_1$$

hence $ln\, L_2 = ln\, a_1 \quad \Rightarrow \quad L_2 = a_1$ **Ans.**

(3) $\lim_{x\to-\infty} F(x) = L_3$ (say)

$$\therefore \quad ln\, L_3 = \lim_{x\to-\infty} \frac{\left(p_1\, ln\, a_1\, a_1^x + p_2\, ln\, a_2\, a_2^x + ... + p_n\, ln\, a_n\, a_n^x\right)}{p_1 a_1^x + p_2 a_2^x + ... + p_n a_n^x}$$

dividing by $(a_n)^x$ and taking $\lim_{x\to-\infty},\ \left(\frac{a_1}{a_n}\right)^x,\ \left(\frac{a_2}{a_n}\right)^x$ etc vanishes

$$\therefore \quad ln\, L_3 = \frac{p_n\, ln\, a_n}{p_n} \quad \Rightarrow \quad L_3 = a_n$$

63 Let $f : R^+ \to R$ be a differentiable function with $f(1) = 3$ and satisfying :

$$\int_1^{xy} f(t)\,dt = y\int_1^x f(t)\,dt + x\int_1^y f(t)\,dt\ ;\ \forall\ \ x, y \in R^+$$

then find $f(x)$.

Sol We have $\int_1^{xy} f(t)\,dt = y\int_1^x f(t)\,dt + x\int_1^y f(t)\,dt$

Differentiating both sides w.r.t. x treating y as constant; we get

$$f(xy).y = yf(x) + \int_1^y f(t)\,dt$$

Putting $x = 1$, we get $yf(y) = yf(1) + \int_1^y f(t)\,dt$

$$\Rightarrow \quad yf(y) = 3y + \int_1^y f(t)\,dt \qquad (\because f(1) = 3)$$

Again differentiating both sides w.r.t. y, we get

$$yf'(y) + f(y).1 = 3 + f(y)$$

$$\Rightarrow \quad f'(y) = \frac{3}{y}$$

Integrating both sides w.r.t. y with limit 1 to x then

$$yf'(1) = 3\ln x - 3\ln 1$$

$$f(x) - f(1) = 3\ln x - 3\ln 1$$

$$\Rightarrow \quad f(x) - 3 = 3\ln x - 0 \qquad (\because f(1) = 3)$$

$\Rightarrow \quad f(x) = 3 + 3\ln x$

$= 3\ln e + 3\ln x = 3\ln(ex)$

Hence $f(x) = 3\ln(ex)$.

64 Let $f(x^m y^n) = mf(x) + nf(y) \ \forall \ x, y \in R^+$ and $\forall \ m, n \in R$. If $f'(x)$ exists and has the value $\frac{e}{x}$, then find $\lim_{x\to 0}\frac{f(1+x)}{x}$.

Sol $\because \quad f(x^m y^n) = mf(x) + nf(y)$ ....(1)

Putting $x = y = m = n = 1$, then $f(1) = f(1) + f(1)$

$\Rightarrow \quad f(1) = 0$

$$\therefore \quad f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right) - f(x\cdot 1)}{h}$$

$$= \lim_{h\to 0}\frac{f\left\{\left(x^{1/m}\right)^m\left\{\left(1+\frac{h}{x}\right)^{1/n}\right\}^n\right\} - f\left\{\left(x^{1/m}\right)^m\left((1)^{1/n}\right)^n\right\}}{h}$$

$$= \lim_{h\to 0}\frac{mf\left(x^{1/m}\right) + nf\left\{\left(1+\frac{h}{x}\right)^{1/n}\right\} - mf\left(x^{1/m}\right) - nf(1)}{h}$$

$$= \lim_{h\to 0}\frac{nf\left\{\left(1+\frac{h}{x}\right)^{1/n}\right\}}{h}$$

$$= \lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{x\left(\frac{h}{x}\right)} \qquad \left(\text{Putting } y = 1 \text{ in (1) then } f(x^m) = mf(x)\right)$$

$$\Rightarrow \quad \frac{e}{x} = \frac{1}{x}\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{\left(\frac{h}{x}\right)} \qquad \Rightarrow \quad \lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{\left(\frac{h}{x}\right)} = e$$

Hence $\lim_{h\to 0}\frac{f(1+x)}{x} = e$

65 Let f be a continuous and differentiable function in (x_1, x_2). If $f(x).f'(x) \geq x\sqrt{1-(f(x))^4}$

and $\lim_{x\to x_1^+}(f(x))^2 = 1$ and $\lim_{x\to x_2^-}(f(x))^2 = \frac{1}{2}$ for $x \in (x_1, x_2)$, then prove that $x_1^2 - x_2^2 \geq \frac{\pi}{3}$

(assume that $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$ holds everywhere).

Sol Given $f(x)\cdot f'(x) \geq x\sqrt{1-(f(x))^4}$

$\Rightarrow \quad \frac{f(x)f'(x)}{\sqrt{1-(f(x))^4}} - x \geq 0 \qquad$ or $\qquad \frac{2f(x)f'(x)}{\sqrt{1-(f(x))^4}} - 2x \geq 0$

or $\quad \frac{d}{dx}\left\{\sin^{-1}(f(x))^2 - x^2\right\} \geq 0$

$\Rightarrow \quad F(x) = \sin^{-1}(f(x))^2 - x^2$ is a non decreasing function.

$\Rightarrow \quad \lim_{x\to x_1^+} F(x) \leq \lim_{x\to x_2^-} F(x)$

$\Rightarrow \quad \lim_{x\to x_1^+}\left\{\sin^{-1}(f(x))^2 - x^2\right\} \leq \lim_{x\to x_2^-}\left\{\sin^{-1}(f(x))^2 - x^2\right\}$

$\Rightarrow \quad \frac{\pi}{2} - x_1^2 \leq \frac{\pi}{6} - x_2^2 \qquad \Rightarrow \qquad x_1^2 - x_2^2 \geq \frac{\pi}{3}$.

66 Are there any non-constant differentiable functions $f: R \to R$ such that $f(f(f(x))) = f(x) \geq 0 \ \forall \ x \in R$?

Sol Given $f(f(f(x))) = f(x)$ ....(1)

Applying f to both sides of the equation (1), then

$$f(f(f(x))) = f\{f(x)\} \qquad ....(2)$$

If $g(x) = f(f(x)) \ \forall \ x \in R$ then equation (2) can be written as $g(g(x)) = g(x)$; g is also a differentiable function on R and $g(x) \geq 0 \ \forall \ x \in R$.

Then the range $T = g(R)$ of g is an interval in $[0, \infty)$. Let a be the infimum of T.

Since $g(t) = t$ for all $t \in T$ and g is continuous.

$\Rightarrow \quad g(a) = a$

Assume T has more than one element. Choose $\delta > 0$ such that $(a, a+\delta \subseteq T)$.

Then $x \in (a-\delta, a)$

$\Rightarrow \quad g(x) \geq g(a) = a \qquad \therefore \qquad \frac{g(x)-g(a)}{x-a} \leq 0$

$\therefore \quad Lg'(a) = \lim_{x\to a-}\frac{g(x)-g(a)}{x-a} \leq 0$

$$= \lim_{h\to 0}\frac{g(a-h)-g(a)}{-h} \leq 0 \qquad ....(3)$$

For $x \in (a, a+\delta)$ we have $$\frac{g(x)-g(a)}{x-a} = 1$$

Hence $$Rg'(a) = \lim_{x \to a+} \frac{g(x)-g(a)}{x-a} = 1 \qquad ....(4)$$

As g is differentiable at a, therefore (3) and (4) are contradictory. This concludes that T is a single point i.e., g is a constant function,

$$g(x) = c \quad \forall \quad x \in R, \qquad \text{(c is constant)}$$

from (1), $$f(c) = f(x) \quad \forall \quad x \in R$$

This shows that f is a constant function. Thus there is no non-constant differentiable function satisfying (1).

67 Let $f(x) = x^3 - 3x^2 + 6 \quad \forall \quad x \in R$ and

$$g(x) = \begin{cases} \max\{f(t) : x+1 \le t \le x+2, -3 \le x < 0\} \\ 1-x, \qquad \text{for} \qquad x \ge 0 \end{cases}$$

Test continuity of $g(x)$ for $x \in [-3, 1]$.

Sol Since $f(x) = x^3 - 3x^2 + 6$

$\Rightarrow \quad f'(x) = 3x^2 - 6x$

$= 3x(x-2)$

for maximum and minima $f'(x) = 0$

$\therefore \quad x = 0, 2$

$f''(x) = 6x - 6$

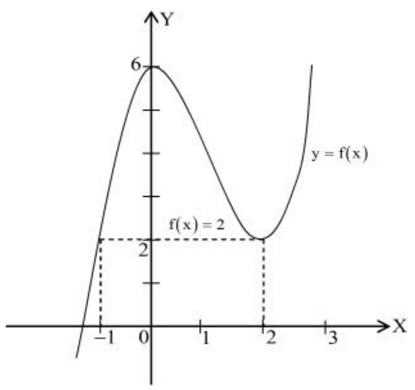
$f''(0) = -6 < 0$ (local maxima at $x = 0$)

$f''(2) = 6 > 0$ (local minima at $x = 2$)

Cut off x-axis $x^3 - 3x^2 + 6 = 0$ has maximum 2 positive and 1 negative real roots.

Cut off y-axis. $F(0) = 6$.

Now graph of $f(x)$ is :



Clearly f(x) is increasing in $(-\infty, 0) \cup (2, \infty)$ and decreasing in (0,2)

$\Rightarrow \quad x + 2 < 0 \quad \Rightarrow \quad x < -2 \quad \Rightarrow \quad -3 \le x < -2$

$\Rightarrow \quad -2 \le x + 1 < -1 \quad$ and $\quad -1 \le x + 2 < 0$

in both cases f(x) increases (maximum) of $g(x) = f(x+2)$

$\therefore \quad g(x) = f(x+2); -3 \le x < -2 \quad$ ...(1)

and if $x + 1 < 0$ and $0 \le x + 2 < 2 \quad \Rightarrow \quad -2 \le x < -1$

then $g(x) = f(0)$

Now for $x + 1 \ge 0$ and $x + 2 < 2 \quad \Rightarrow \quad -1 \le x < 0, g(x) = f(x+1)$

Hence $$g(x) = \begin{cases} f(x+2) & ; \; -3 \le x < -2 \\ f(0) & ; \; -2 \le x < -1 \\ f(x+1) & ; \; -1 \le x < -0 \\ 1 - x & ; \; x \ge 0 \end{cases}$$

Hence g(x) is continuous in the interval $[-3, 1]$.

68 $f : [0, 1] \to R$ is defined as $f(x) = \begin{bmatrix} x^3(1-x)\sin\left(\frac{1}{x^2}\right) & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{bmatrix}$, then prove that

(a) f is differentiable in [0, 1] (b) f is bounded in [0, 1] (c) f' is bounded in [0, 1]

Sol. $$f(x) = \begin{bmatrix} x^3(1-x)\sin\left(\frac{1}{x^2}\right) & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{bmatrix}$$

$$f'(0^+) = \lim_{h \to 0} \frac{h^3(1-h)\sin\frac{1}{h^2} - 0}{h} = 0$$

$$f'(1^-) = \lim_{h \to 0} \frac{(1-h)^3(+h)\sin\frac{1}{(1-h)^2} - 0}{-h} = \lim_{h \to 0} -(1-h)^3 \sin\frac{1}{(1-h)^2} = -\sin 1$$

Hence f is derivable in [0, 1], obviously f is continuous in [0, 1] hence f is bounded

hence $$f'(x) = \begin{bmatrix} (x^3 - x^4)\cos\left(\frac{1}{x^2}\right)\left(-\frac{2}{x^3}\right) + \sin\frac{1}{x^2}(3x^2 - 4x^3) & x \ne 0 \\ 0 \quad \text{if } x = 0 \end{bmatrix}$$

$\lim_{x \to 1^-} = (0) + \sin 1(3 - 4)$, hence f' is also bounded.

69 Discuss the continuity of f in [0,2] where $f(x) = \begin{cases} |4x-5|\,[x] & \text{for } x > 1 \\ [\cos \pi x] & \text{for } x \le 1 \end{cases}$; where [x] is the greatest integer not greater than x.

Sol. $$f(x) = \begin{cases} |4x-5|\,[x] & \text{for } 1 < x \le 2 \\ [\cos \pi x] & \text{for } 0 \le x \le 1 \end{cases} = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } 0 < x \le \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} < x \le 1 \\ (5-4x) & \text{if } 1 < x < \frac{5}{4} \\ (4x-5) & \text{if } \frac{5}{4} \le x < 2 \\ 6 & \text{if } x = 2 \end{cases}$$

Clearly f(x) is discont. for x = 0, 1/2, 1 & 2.

70 If f(x) = x + {-x} + [x] , where [x] is the integral part & {x} is the fractional part of x. Discuss the continuity of f in [– 2, 2].

Sol. $f(x) = x + \{-x\} + [x]$

if $n < x < n+1$, then $f(x) = 2n+1$

$\{$as for nonintegral values $\{-x\} = 1 - x + [x]$ and $[x] = n\}$

if $x = n$, then $f(x) = 2n$

Hence $f(x) = \begin{cases} 2n & \text{if } x = n \\ 2n+1 & \text{if } n < x < n+1 \\ 2n+2 & \text{if } x = n+1 \end{cases}$