CONTINUITY & DIFFERENTIABILITY
EXERCISE 1(C)

1. $\underset{h\to 0}{\text{Limit}}\ g(n+h) = \underset{h\to 0}{\text{Limit}}\ \dfrac{e^h - \cos 2h - h}{h^2}$

$$= \underset{h\to 0}{\text{Limit}}\ \frac{e^h - h - 1}{h^2} + \underset{h\to 0}{\text{Limit}}\ \frac{(1-\cos 2h)}{4h^2}.4 = \frac{1}{2} + 2 = \frac{5}{2}$$

$$\underset{h\to 0}{\text{Limit}}\ g(n-h) = \frac{e^{1-\{n-h\}} - \cos 2(1-\{n-h\}) - (1-\{n-h\})}{(1-\{n-h\})^2}$$

$$= \underset{h\to 0}{\text{Lim}}\ \frac{e^h - \cos 2h - h}{h^2}\ \left(\{n-h\} = \{-h\} = 1-h\right) = \frac{5}{2}$$

$g(n) = \dfrac{5}{2}$. Hence g(x) is continuous at $\forall\, x \in I$.

Hence g (x) is continuous $\forall\, x \in R$]

2. By theorem, if g and h are continuous functions on the open interval (a, b), then g/h is also continuous at all x in the open interval (a, b) where h (x) is not equal to zero.

3. $$h(x) = \begin{cases} \dfrac{2\cos x - \sin 2x}{(\pi - 2x)^2} & x < \dfrac{\pi}{2} \\[2ex] \dfrac{e^{-\cos x} - 1}{8x - 4\pi} & x > \dfrac{\pi}{2} \end{cases}$$

LHL at $x = \pi/2$

$$\underset{h\to 0}{\text{Lim}}\ \frac{2\sin h - \sin 2h}{4h^2} = \underset{h\to 0}{\text{Lim}}\ \frac{2\sin h(1-\cos h)}{4h^2} = 0$$

RHL: $$\underset{h\to 0}{\text{Lim}}\ \frac{e^{\sin h} - 1}{\left((\pi/2)+h\right) - 4\pi} = \underset{h\to 0}{\text{Lim}}\ \frac{e^{\sin h} - 1}{8h} \cdot \frac{\sin h}{\sin h} = \frac{1}{8}$$

$\Rightarrow$ h (x) is discontinuous at $x = \pi/2$.
Irremovable discontinuity at $x = \pi/2$.

$$f\left(\frac{\pi}{2}^+\right) = 0 \text{ and } g\left(\frac{\pi}{2}^-\right) = \frac{1}{8} \quad \Rightarrow \quad f\left(\frac{\pi}{2}^+\right) \neq g\left(\frac{\pi}{2}^-\right)\ ]$$

4. $\underset{x\to 0}{\text{Lim}}\ \dfrac{x - e^x + 1 - (1-\cos 2x)}{x^2} = -\dfrac{1}{2} - 2 = -\dfrac{5}{2}$; hence for continuity $f(0) = -\dfrac{5}{2}$

$\therefore\quad [f(0)] = -3$; $\{f(0)\} = \left\{-\dfrac{5}{2}\right\} = \dfrac{1}{2}$; hence $[f(0)]\ \{f(0)\} = -\dfrac{3}{2} = -1.5$

5. $g'(0^+) = \underset{h\to 0}{\text{Lim}}\ \dfrac{\cosh - 1}{h} = 0$

$g'(0^-) = \lim_{h\to 0} \frac{-h+b-1}{-h}$ for existence of limit b = 1 thus $g'(0^-) = 1$

Hence g can not be made differentiable for any value of b.]

6. $\left[\begin{array}{l} x - 2k\pi \text{ for } 2k\pi - \frac{\pi}{2} \le x \le 2k\pi + \frac{\pi}{2} \\ (2k+1)\pi - x \text{ for } 2k\pi + \frac{\pi}{2} \le x < 2k\pi + \frac{3\pi}{2} \end{array}\right.$

7. $f(1^+) = f(1^-) = f(1) = 2$

$f(0) = 1, \quad f(2) = 2$

$f(2^-) = 1; \quad f(2) = 2 \quad \Rightarrow \quad$ f is not continuous at x = 2

8. $x^{1/3}$ is not differentiable at x = 0]

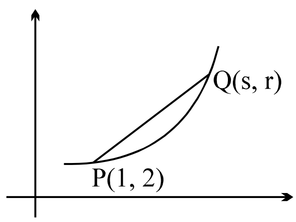
9. $f(2^+) = 8 \;;\; f(2^-) = 16$

10. **I** By definition $f'(1)$ is the limit of the slope of the secant line when $s \to 1$.

Thus $f'(1) = \lim_{s\to 1} \frac{s^2+2s-3}{s-1}$

$= \lim_{s\to 1} \frac{(s-1)(s+3)}{s-1}$

$= \lim_{s\to 1}(s+3) = 4 \quad \Rightarrow \quad$ (D)



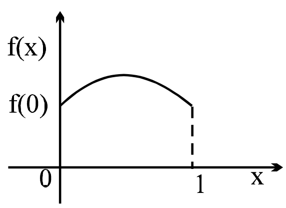
II By substituting x = s into the equation of the secant line, and cancelling by s – 1 again, we get $y = s^2 + 2s - 1$. This is f (s), and its derivative is $f'(s) = 2s + 2$, so $f'(1) = 4$.]

11. $g(x) = x - [x] = \{x\}$

f is continuous with f(0) = f(1)

$h(x) = f(g(x)) = f(\{x\})$

Let the graph of f is as shown in the figure satisfying



$f(0) = f(1)$

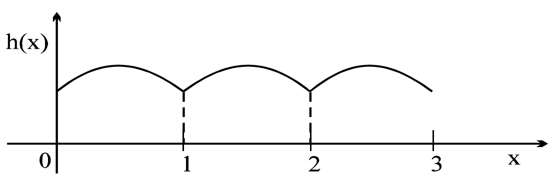
now $h(0) = f(\{0\}) = f(0) = f(1)$

$h(0.2) = f(\{0.2\}) = f(0.2)$

$h(1.5) = f(\{1.5\}) = f(0.5)$ etc.

Hence the graph of h(x) will be periodic graph as shown

$\Rightarrow$ h is continuous in R $\Rightarrow$ C



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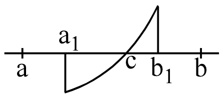
12. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(h)+|x|h+xh^2}{h}$ where x = h and y = x

$\because$ f(0) = 0; hence $f'(x) = \lim_{h\to 0}\left(\frac{f(h)-f(0)}{h} + |x| + xh\right)$

$f'(x) = f'(0) + |x| = |x|$

13. I and II are false. The function f(x) = 1/x, 0 < x < 1, is a counter example.
Statement III is true. Apply the intermediate value theorem to f on the closed interval $[a_1, b_1]$



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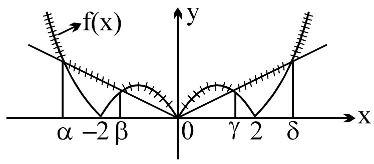
14. $f(x)$ is non differentiable at x = α, β, 0, γ, δ

and $g(x)$ is non differentialbe at x = α, β, 0, – 2, 2 ⇒ (B)



15. $g(x) = \begin{cases} 3x^2 - 4\sqrt{x} + 1 & \text{for } x < 1 \\ ax + b & \text{for } x \ge 1 \end{cases}$

for differentiability at x = 1, $g'(1^+) = g'(1^{-1})$

$$a = 6x - \frac{4}{2\sqrt{x}} \Rightarrow \quad a = 6 - 2 = 4$$

for continuity at x = 1, $g(1^+) = g(1^-)$

$$a + b = 3 - 4 + 1 \quad \Rightarrow \quad a + b = 0 \quad \Rightarrow \quad b = -4$$

a = 4, and b = – 4 Ans.]

16. $g(x) = \frac{\sin\frac{\pi[x]}{4}}{[x]}$

obv. cont. at x = 3/2

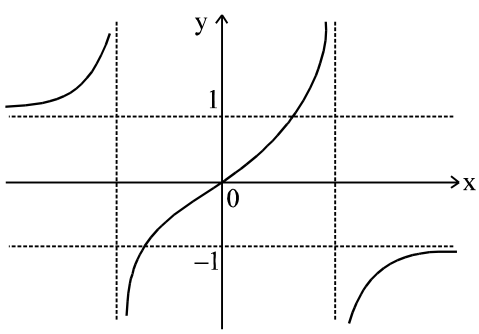
at x = 2 $f(2^-) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f(2) = \frac{\sin\frac{\pi}{2}}{2} = \frac{1}{2}$

Hence discontinuous at x = 2

17. not derivable at x = 0 and 2

18. $f(x) = \begin{cases} \frac{x}{1-x} & \text{if } x \ge 0,\ x \ne 1 \\ \frac{x}{1+x} & \text{if } x < 0,\ x \ne -1 \end{cases}$



and $f'(x) = \begin{bmatrix} \frac{1}{(1-x)^2} & \text{if } x > 0,\ x \neq 1 \\ \frac{1}{(1+x)^2} & \text{if } x < 0,\ x \neq -1 \end{bmatrix}$]

19. $f(-1) = b(1-1) + 1 = 1$

and $\lim_{h\to 0} f(-1+h) = 1$

$\lim_{h\to 0} f(-1-h) = \sin((-1+h+a)\pi) = -\sin \pi a$

for continuity $\sin \pi a = -1 = \sin\left(2n\pi + \frac{3\pi}{2}\right) \Rightarrow \pi a = 2n\pi + \frac{3\pi}{2} \Rightarrow a = 2n + \frac{3}{2}$

hence $a = 2n + \frac{3}{2}$, $n \in I$ and $b \in R$]

20. $f(0) = \lim_{x\to 0} \frac{\ell n\left(e^{x^2} + 2\sqrt{x}\right)}{\sqrt{x}} = \lim_{x\to 0} \ln\left(e^{x^2} + 2\sqrt{x}\right)^{\frac{1}{\sqrt{x}}}$

$= \lim_{x\to 0} \frac{1}{\sqrt{x}}\left(e^{x^2} + 2\sqrt{x} - 1\right) = \lim_{x\to 0}\left(\frac{e^{x^2} - 1}{\sqrt{x}} + 2\right) = 2$

21. $f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = 1$ also $f(0) = -c$

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x) + f(h) + c - f(x)}{h}$

$= \lim_{h\to 0} \frac{f(h) - f(0)}{h} = f'(0) = 1$

$\therefore$ $f'(x) = 1$

22. Differentiate column wise, where $\Delta_1 = -4$; $\Delta_2 = 0$ and $\Delta_3 = 8$

23. $D^*f(x) = 2f(x).f'(x)$
$D^*(x\, l\text{n}x) = 2x\, l\text{n}x\,(1 + l\text{n}x)$]

24. In the immediate neighborhood of $x = \pi/2$, $\sin x > \sin^3 x \Rightarrow |\sin x - \sin^3 x| = \sin x - \sin^3 x$

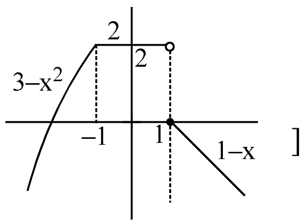
Hence for $x \neq \pi/2$, $f(x) = \left[\frac{2(\sin x - \sin^3 x) + \sin x - \sin^3 x}{2(\sin x - \sin^3 x) - \sin x + \sin^3 x}\right] = \frac{3\sin x - 3\sin^3 x}{\sin x - \sin^3 x} = 3$

Hence f is continuous and diff. at $x = \pi/2$

25. $y = (A + Bx)e^{mx} + (m-1)^{-2} \cdot e^x$
$y \cdot e^{-mx} = (A + Bx) + (m-1)^{-2} \cdot e^{(1-x)x}$
$e^{-mx} \cdot y_1 - my + e^{-mx} = B - (m-1)^{-1} \cdot e^{-(m-1)x}$
$e^{-mx} \cdot y_2 - y_1 e^{-mx} \cdot m - m[e^{-mx} \cdot y_1 - y\, e^{-mx} \cdot m] = e^{-(m-1)x}$

$e^{-mx} \cdot y_1 - m_2 y_1 e^{-mx} + my \cdot e^{-mx} = e^{-(m-1)x}$
$y_2 - 2my_1 + my = e^x$ Ans.]

26. Let $f(x) = px^2 + qx + r$
$f(1) = f(-1)$ gives $p + q + r = p - q + r$
hence $q = 0$
Hence $f(x) = px^2 + r$
$f'(x) = 2px$ ....(1)
Given a, b, c are in A.P.
hence 2pa, 2pb, 2pc will also be in A.P.
or $f'(a), f'(b), f'(c)$ will also be in A.P. $\Rightarrow$ (D)]

27. 

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28. $2x + 2yy' = 0$

$x + yy' = 0 \quad \Rightarrow \quad y' = -\frac{x}{y}$ ....(1)

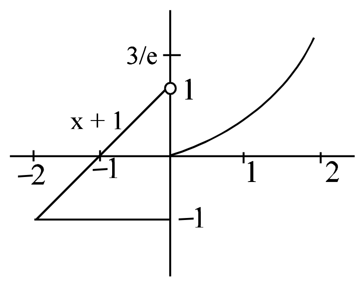
$1 + yy'' + (y')^2 = 0$

$y'' = -\frac{1+(y')^2}{y}$

now $k = \frac{y''}{\left(1+(y')^2\right)^{3/2}} = -\frac{1+(y')^2}{y\left(1+(y')^2\right)^{3/2}} = -\frac{1}{y\sqrt{1+(y')^2}} = -\frac{1}{y\sqrt{1+\frac{x^2}{y^2}}} = -\frac{1}{\sqrt{y^2+x^2}} = -\frac{1}{R}$

29. $f(x) = \begin{cases} (x+1)e^{-2/x} & \text{if } x > 0 \\ x+1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$

the graph of f (x) is
hence f can assume all values for f (– 2) to f (2)



30. Put $\cos\phi = \frac{2}{\sqrt{13}}$; $\sin\phi = \frac{3}{\sqrt{13}}$; $\tan\phi = \frac{3}{2}$

$y = \cos^{-1}\{\cos(x+\phi)\} + \sin^{-1}\{\cos(x-\phi)\}$

$= \cos^{-1}\{\cos(x+\phi) + \frac{\pi}{2} - \cos^{-1}\{\cos(\phi - x)\}$ (think !)

$= x + \phi + \frac{\pi}{2} - \phi + x$

$y = 2x + \frac{\pi}{2}$; $z = \sqrt{1+x^2}$

now compute $\frac{dy}{dz}$

31. We have $f(x) = \frac{x^2 - x}{x^2 + 4x}$

f (x) is not defined at x = 0, – 4

$\therefore$ domain of f = R – {0, – 4}

For all x $\in$ domain of f, we have

$$f(x) = \frac{x^2 - x}{x^2 + 4x} = \frac{x-1}{x+4} = 1 - \frac{5}{x+4}$$

$$f\left(f^{-1}(x)\right) = x$$

$$1 - \frac{5}{f^{-1}(x)+4} = x \quad \text{or} \quad 1 - x = \frac{5}{f^{-1}(x)+4}$$

$$\therefore \quad f^{-1}(x) = \frac{5}{1-x} - 4$$

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{5}{(1-x)^2}; \qquad \therefore \quad \frac{d}{dx}\left(f^{-1}(x)\right)_{\text{at } x=2} = \frac{5}{(1-2)^2} = 5 \text{ Ans.]}$$

32. $f(x) = \sqrt{x + 2\sqrt{2x-4}} + \sqrt{x - 2\sqrt{2x-4}}$

$\therefore$ $f(x) = \sqrt{\left(\sqrt{x-2}+\sqrt{2}\right)^2} + \sqrt{\left(\sqrt{x-2}-\sqrt{2}\right)^2} = \left|\sqrt{x+2}+\sqrt{2}\right| + \left|\sqrt{x-2}-\sqrt{2}\right|$

for $\sqrt{x-2}$ to exist $x \ge 2$

Also, $\sqrt{x-2} + \sqrt{2} > 0$ (always true, think ! why?)

but $\sqrt{x-2} - \sqrt{2} \ge 0$ only if $x \ge 4$

< 0 only if $x < 4$

$\therefore$ now f (x) becomes

$f(x) = \sqrt{x-2} + \sqrt{2} - \sqrt{x-2} + \sqrt{2}$ for $2 \le x < 4$

$= \sqrt{x-2} + \sqrt{2} + \sqrt{x-2} - \sqrt{2}$ for $x \ge 4$

$\therefore$ $f(x) = 2\sqrt{2}$, for $2 \le x < 4$

$= 2\sqrt{x-2}$, for $4 \le x < \infty$

$\because$ f is continuous $[2, 4) \cup [4, \infty)$(verify)

$\therefore$ $f'(x) = 0$, $2 \le x < 4$

$= \frac{1}{\sqrt{x-2}}$, $4 \le x < \infty$

$\therefore$ $f'(102^+) = \frac{1}{\sqrt{102-2}} = \frac{1}{10}$

$\therefore$ $10 f'(102^+) = 1$

33. A : $2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$

$\therefore$ $y'(\sqrt{2}) = -1 = A$

B : $\cos y \cdot y' + \cos x = \sin x \cdot \cos y \cdot y' + \sin y \cdot \cos x$
when $x = y = \pi$
$-y' - 1 = 0 + 0 \quad \Rightarrow \quad y'(\pi) = -1$
C : $2e^{xy}(xy' + y) + e^x e^y y' + e^y e^x - e^x - e^y y' = e \cdot e^{xy}(xy' + y)$
at $x = 1, y = 1$
$2e(y' + 1) + e^2 y' + e^2 - e - ey' = e^2(y' + 1)$
$ey' + e = 0 \quad \Rightarrow \quad y' = -1$
hence $A + B + C = -3$

34. $C_2 \to C_2 - xC_3$
$\Rightarrow f(x) = x^2 (\tan x - \cos x)]$
$\Rightarrow f'(x) = (\tan x - \cos x)\, 2x - x^2 (\sec^2 x + \sin x)$

35. $\frac{x+a}{2} = b \cot^{-1}(b \ln y)$; $\quad \cot\left(\frac{x+a}{2b}\right) = b\, ln\, y$

$\therefore \quad -\operatorname{cosec}^2\left(\frac{x+a}{2b}\right)\frac{1}{2b} = \frac{b}{y}y'$; $\quad \therefore \quad -\frac{1}{2b^2}\left(1+\cot^2\left(\frac{x+a}{2b}\right)\right) = \frac{y'}{y}$

$\therefore \quad -\frac{1}{2b^2}\left((1+(b\, ln\, y))^2\right) = \frac{y'}{y}$; $\quad \therefore \quad -\frac{1}{2b^2}\left(2(b\, ln\, y)\frac{b}{y}y'\right) = \frac{yy''-y'^2}{y^2}$

$\therefore \quad -ln\, y\, y' = y\, y'' - y'^2$; $\quad \therefore \quad y\, y'' = y'^2 - y'\, y\, ln\, y$

$\therefore \quad y\, y'' + y\, y'\, ln\, y = y'^2 - y'\, y\, ln\, y + y\, y'\, ln\, y = y'^2$

36. $\lim\limits_{x\to 0} x^x = 1$; let $l = x^{x^x}$ hence as $x \to 0$, $x^x \to 1$
$\therefore \quad L = (0)' - 1 = -1 \quad \Rightarrow \quad$ (C)

37. $2\frac{d}{dx}(y^3 y_2) = 2(y^3.y_3 + 3y^2 y_1 y_2)$.
Now differentiate $y^2 = P(x)$ thrice)

38. for continuity $\quad \lim\limits_{x\to 0}\frac{1-e^x}{x} = f(0)$; $\quad$ hence $\quad f(0) = -\lim\limits_{h\to 0}\frac{e^x-1}{-x} = -1$

$$f'(0^+) = \lim_{h\to 0}\frac{\frac{1-e^h}{h}+1}{h} = \lim_{h\to 0}\frac{1-e^h+h}{h^2} = \frac{1-h-\left[1+\frac{h}{1!}+\frac{h^2}{2!}+......\right]}{h^2} = -\frac{1}{2}$$

$$f'(0^-) = \lim_{h\to 0}\frac{\frac{1-e^{-h}}{-h}+1}{-h} = \lim_{h\to 0}\frac{1-e^{-h}-h}{h^2} = \frac{1-h-\left[1-\frac{h}{1!}+\frac{h^2}{2!}-......\right]}{h^2} = -\frac{1}{2}$$

$$\text{hence } f(x) = \begin{cases} \frac{1-e^{-x}}{x} & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$$

39. $(a^2-2a-15)e^{ax}+(b^2-2b-15)e^{bx}=0$

$\Rightarrow$ $(a^2-2a-15)=0$ and $b^2-2b-15=0$

$\Rightarrow$ $(a-5)(a+3)=0$ and $(b-5)(b+3)=0$

$\Rightarrow$ $a=5$ or -3 and $b=5$ or -3

$\therefore$ $a\neq b$ hence $a=5$ and $b=-3$

or $a=-3$ and $b=5$

$\Rightarrow$ $ab=-15$ Ans.]

40. When $x=0$, $e^y=e \Rightarrow y=1$

Differentiating w.r.t. x, we get

$$e^y\frac{dy}{dx}+y+x\frac{dy}{dx}=0 \qquad ....(1)$$

$$e^y\frac{d^2y}{dx^2}=e^y\left(\frac{dy}{dx}\right)^2+\frac{dy}{dx}+\frac{dy}{dx}+x\frac{d^2y}{dx^2}=0 \qquad ....(2)$$

When $x=0, y=1$ $\quad\therefore$ From (1) $\frac{dy}{dx}=-\frac{1}{e}$

Putting the data in (2), we get

$$e\cdot\frac{d^2y}{dx^2}+e\cdot\frac{1}{e^2}-\frac{2}{e}=0 \qquad \therefore \frac{d^2y}{dx^2}=-\frac{1}{e^2}$$]

41. $N^r=\cos6x+(1+5)\cos4x+(5+10)\cos2x+10$

$=\cos6x+\cos4x+5(\cos4x+\cos2x)+10(1+\cos2x)$

$=2\cos5x\ \cos x+10\cos3x\ \cos x+20\cos^2x$

$=2\cos x\,[\cos5x+5\cos3x+10\cos x]$

------Denominator------

$$\therefore\ y=\frac{N^r}{D^r}=2\cos x \qquad \therefore \frac{dy}{dx}=-2\sin x \quad \Rightarrow \text{(C)}$$

42. $y^4=x^2-6$

$$4y^3\frac{dy}{dx}=2x \quad\Rightarrow\quad y^3\frac{dy}{dx}=\frac{x}{2} \quad\Rightarrow\quad y^3\frac{d^2y}{dx^2}+3y^2\left(\frac{dy}{dx}\right)^2=\frac{1}{2}$$

$$y^3\frac{d^2y}{dx^2}+3y^2\left(\frac{x}{2y^3}\right)^2=\frac{1}{2} \quad\Rightarrow\quad y^3\frac{d^2y}{dx^2}+3y^2\frac{x^2}{4y^6}=\frac{1}{2} \quad\Rightarrow\quad y^3\frac{d^2y}{dx^2}+\frac{3x^2}{4y^4}=\frac{1}{2}$$

$$\Rightarrow\quad y^3\frac{d^2y}{dx^2}=\frac{1}{2}-\frac{3x^2}{4y^4} \quad\Rightarrow\quad y^3\frac{d^2y}{dx^2}=\frac{2y^4-3x^2}{4y^4} \quad\Rightarrow\quad \frac{d^2y}{dx^2}=\frac{2y^4-3x^2}{4y^7}$$

43. $$\frac{d}{dx}\{[f(x)]^2-[\phi(x)]^2\}=2\,[f(x)\cdot f'(x)-\phi(x)\cdot\phi'(x)]=2\,[f(x)\cdot\phi(x)-\phi(x)\cdot f(x)]=0$$

[$\because$ $f'(x)=\phi(x)$ and $\phi'(x)=f(x)$]

$\Rightarrow$ $[f(x)]^2-[\phi(x)]^2=$ constant

$\therefore$ $[f(10)]^2-[\phi(10)]^2=[f(3)]^2-[\phi(3)]^2=[f(3)]^2-[f'(3)]^2=25-16=9$

44. $$(x+y)\left(\frac{\cos^4\alpha}{x}+\frac{\sin^4\alpha}{y}\right)=1=(\cos^2\alpha+\sin^2\alpha)^4$$

$\therefore \quad \frac{y}{x}\cos^4\alpha + \frac{x}{y}\sin^4\alpha - 2\sin^2\alpha\cos^2\alpha = 0$

or $\quad \left(\sqrt{\frac{y}{x}}\cos^2\alpha - \sqrt{\frac{x}{y}}\sin^2\alpha\right)^2 = 0$

$\therefore \quad \tan^2\alpha = \frac{y}{x}$ or $y = x\tan^2\alpha \qquad \therefore \frac{dy}{dx} = \tan^2\alpha$

45. $y = e^x + x$; diff. w.r.t y,

$1 = (e^x + 1)\frac{dx}{dy}$; $\frac{dx}{dy} = \frac{1}{e^x + 1}$

$\Rightarrow \left.\frac{dx}{dy}\right]_{x=\ell n2} = \frac{1}{e^{\ell n2} + 1} = \frac{1}{3}$

46. $\frac{dy}{dx} = \frac{x}{1+x^2} + \tan^{-1}x + \frac{1}{\left|\frac{1}{x}\right|\sqrt{\frac{1}{x^2} - 1}}\left(-\frac{1}{x^2}\right)$

$= \frac{x}{1+x^2} + \tan^{-1}x + \frac{|x|^2}{\sqrt{1-x^2}}\left(-\frac{1}{x^2}\right)$

as $x \to 0$, $\frac{dy}{dx} = -1$ Ans.

<u>Alternatively</u>:

$f'(0^+) = \lim_{h\to 0}\frac{h\tan^{-1}(h) + \sec^{-1}(1/h) - \pi/2}{h} = \lim_{h\to 0}\frac{\cos^{-1}(h) - \pi/2}{h}$

$= \lim_{h\to 0}\frac{-\sin^{-1}(h)}{h} = -1$

Similarly $\quad f'(0^-) = -1$

Hence $f'(0) = -1$

47. $y = \cfrac{\sin x}{1 + \cfrac{\cos x}{1 + \cfrac{\sin x}{1 + \cdots}}}$; $y = \cfrac{\sin x}{1 + \cfrac{\cos x}{1+y}} = \frac{\sin x(1+y)}{1 + y + \cos x}$; $\qquad y(0) = 0$

$y(1 + y + \cos x) = \sin x\,(1 + y)$

$y' + 2yy' + \cos x\, y' - y\sin x = \cos x(1 + y) + \sin x y'$

$y'(0)\,[1 + 2y + \cos x] - 0 = 1 + 0$

$2y'(0) = 1 \qquad \Rightarrow \qquad y'(0) = \frac{1}{2}$

48. $f'(0) = \lim_{h\to 0}\frac{e^{-1/h^2}}{h} = \lim_{h\to 0}\frac{1/h}{e^{1/h^2}} = \frac{-1/h^2}{-e^{1/h^2}\cdot\frac{2}{h^3}} = \lim_{h\to 0}\frac{2h^3}{h^2 e^{1/h^2}} = 0$

Hence f is differentiable at x = 0. Also $\lim_{x\to\pm\infty} e^{-\frac{1}{x^2}} \to 1 \Rightarrow C$

Alternatively:

check concavity by finding $\frac{d^2y}{dx^2}$ and eliminate D.

49. Given $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$

now $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ $\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{\frac{dx}{dy}}\right) = \frac{d}{dy}\left(\frac{1}{\frac{dx}{dy}}\right).\frac{dy}{dx} = -\frac{1}{\left(\frac{dx}{dy}\right)^2}.\frac{d^2x}{dy^2}.\frac{1}{\frac{dx}{dy}}$

$$\frac{d^2y}{dx^2} = -\frac{\frac{d^2x}{d^2y}}{\left(\frac{dx}{dy}\right)^3} \qquad \text{(putting in (1))}$$

$$-\frac{\frac{d^2x}{d^2y}}{\left(\frac{dx}{dy}\right)^3} + y\frac{dy}{dx} = 0 \Rightarrow \quad y\left(\frac{dy}{dx}\right)^2 - \frac{d^2x}{dy^2} = 0 \quad \Rightarrow C$$

50. Let $f(x) = y \Rightarrow x = f^{-1}(y) = g(y) \Rightarrow \quad x = e^{e^y}$

$\Rightarrow \quad \frac{dx}{dy} = e^{e^y} \cdot e^y = e^{e^y + y} = g'(y)$

hence $g'(x) = e^{e^x + x}$