CONTINUITY & DIFFERENTIABILITY
EXERCISE 1(B)

1. $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-}(2x-1) = -3$

$f(-1) = (4x+1)_{x=-1} = -3$

$\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+}(4x+1) = -3$

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}(4x+1) = 5$

$f(1) = (4x+1)_{x=1} = 5$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^-}(3-2x) = 1$

As $\lim\limits_{x\to -1^-} f(x) = f(-1) = \lim\limits_{x\to -1^+} f(x) = -3$ Hence f(x) is cont. at x = –1.

Further $\lim\limits_{x\to 1^-} f(x) = f(1) \neq \lim\limits_{x\to 1^+} f(x)$ hence f(x) is discont. at x = 1.

Being all linear functions f(x) is cont. everywhere else. **Ans.[B]**

2. $f(x) = \begin{cases} \dfrac{|\tan x|}{x}, & x<0 \\ \dfrac{1-e^{\sin x}}{\sin x}, & x>0 \end{cases}$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}\dfrac{|\tan x|}{x} = \lim\limits_{x\to 0^-}\dfrac{-\tan x}{x} = -1$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-}\dfrac{1-e^{\sin x}}{\sin x} = 1$

As $\lim\limits_{x\to 0^-} f(x) \neq \lim\limits_{x\to 0^+} f(x)$ hence $f(x)$ is discont. at $x=0$ and discontinuity is not removable.

Ans.[C]

3. As codomain of f(x) is open interval hence f(x) can approach ± 2 as limit but cant acquire these values. If f(x) is onto then at the points where it approaches ± 2 it will be discontinuous but as f(x) is given cont. hence it must be an into function. **Ans.[B]**

4. As codomain of f(x) contains only natural numbers and no two natural numbers are adjacent on number line (between every two consecutive natural numbers there will be infinitely many non integral numbers) hence to be continuous f(x) must be a constant function. **Ans.[B]**

5. f(g(x)) may not be continuous at x = a as if g(a) = b and f(x) is discont. at x = b, then f(g(x) will be discontinuous but it is not neccessary. **Ans.[D]**

6. f(–1–0) = –1, f(–1) = – (–1) = 1

$\Rightarrow$ f(–1–0) ≠ f(–1)

$\Rightarrow$ f(x) is not continuous at x = –1

Further , f(1) = –1

f (1+0) = 1 $\Rightarrow$ f (1) ≠ f(1+0)

$\Rightarrow$ f(x) is not continuous at x = 1. **Ans.[D]**

7. Since f(x) is continuous at x = 0

$\therefore \lim_{x\to 0} f(x) = f(0)$

but f(0)= 0 (given)

$\therefore \lim_{x\to 0} f(x) = \lim_{x\to 0} x^k \cos(1/x)$

$= 0$, if $k > 0$. **Ans.[B]**

8. Obviously function f(x) is discontinuous at x = 0 and x= 1 because the function is not defined, when x< 0 and x> 1, therefore f(0–0) and f(1+0) do not exist. Again

$$f\left(\frac{1}{2}+0\right) = \lim_{x\to 1/2}\left(\frac{3}{2}-x\right) = 1$$

$$f\left(\frac{1}{2}-0\right) = \lim_{x\to 1/2}\left(\frac{1}{2}-x\right) = 0$$

$$\because f\left(\frac{1}{2}+0\right) \neq f\left(\frac{1}{2}-0\right)$$

function f(x) is discontinuous at $x = \frac{1}{2}$ **Ans.[B]**

9. $\because$ f(x) is continuous at x =2

$\therefore f(2-0) = f(2+0) = f(2) = k$

But f(2+0)

$$= \lim_{h\to 0}\frac{(2+h)^3+(2+h)^2-16(2+h)+20}{(2+h-2)^2}$$

$$= \lim_{h\to 0}\frac{h^3+7h^2}{h^2} = 7$$ **Ans. [C]**

10. Since f(x) is continuous at x = 2

$\therefore f(2) = \lim_{x\to 2^+} f(x)$

$\Rightarrow \quad 1 = \lim_{x\to 2^+} (ax+b)$

$\therefore \quad 1 = 2a + b$ ...(1)

Again f(x) is continuous at x = 4,

$\therefore \quad f(4) = \lim_{x\to 4^-} f(x)$

$\Rightarrow 7 = \lim_{x\to 4} (ax+b)$

$\therefore 7 = 4a + b$ ...(2)

Solving (1) and (2), we get a= 3, b = –5. **Ans.[B]**

11. Let us first examine continuity at x = 0.

f (0) = 0 $(\because 0 \in Q)$

$= f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$

$= \lim_{h\to 0} \{-h$ or h according as $-h \in Q$ or $-h \notin Q)$

= 0

$f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$

$= \lim_{h\to 0}\{h \text{ or } -h\} = 0$

$f(0) = f(0-0) = f(0+0)$

$\Rightarrow$ f(x) is continuous at x = 0.
Now let $a \in R$, $a \neq 0$, then

$f(a-0) = \lim_{h\to 0} f(a-h)$

$= \lim_{h\to 0} \{(a-h) \text{ or } -(a-h)\}$

= a or –a, which is not unique.

$\Rightarrow$ f(a–0) does not exist

$\Rightarrow$ f(x) is not continuous at $a \in R_0$.

Hence f(x) is continuous only at x = 0. **Ans.[B]**

12. We know that [x] is discontinuous at every integer. Therefore it is continuous only at x = 1/2, while the function x is continuous at all points x= 0, –1, 1, 1/2. Thus the given function is continuous only at x = 1/2. **Ans.[D]**

13. $$f\left(\frac{\pi}{2}-0\right) = \lim_{h\to 0} \frac{1-\sin^3\left(\frac{\pi}{2}-h\right)}{3\cos^2\left(\frac{\pi}{2}-h\right)}$$

$$= \lim_{h\to 0} \frac{1-\cos^3 h}{3\sin^2 h}$$

$$= \lim_{h\to 0} \frac{(1-\cos h)(1+\cos h+\cos^2 h)}{3(1-\cos h)(1+\cos h)} = 1/2$$

$$f\left(\frac{\pi}{2}+0\right) = \lim_{h\to 0} \frac{b\left[1-\sin\left(\frac{\pi}{2}+h\right)\right]}{\left[\pi-2\left(\frac{\pi}{2}+h\right)\right]}$$

$$= \lim_{h\to 0} \frac{b(1-\cos h)}{4h^2}$$

$$= \lim_{h\to 0} \frac{2b\sin^2 h/2}{4h^2} = \frac{b}{8}$$

Now f(x) is continuous at $x = \frac{\pi}{2}$

$$\Rightarrow f\left(\frac{\pi}{2}-0\right) = f\left(\frac{\pi}{2}+0\right) = f\left(\frac{\pi}{2}\right) \Rightarrow \frac{1}{2} = \frac{b}{8} = a$$

$\therefore$ a = 1/2, b = 4 **Ans.[C]**

14. Obviously the function f(x) is continuous at x = 1 and 3. Therefore $\lim_{x\to 1^+} f(x) = f(1)$

$\Rightarrow a + b = 2$ ...(1)

and $\lim_{x\to 3^-} f(x) = f(3)$

$\Rightarrow 3a + b = 6$ ...(2)

Solving (1) and (2) , we get a = 2, b = 0. **Ans.[C]**

15. $$f(0-0) = \lim_{x\to 0} \frac{1-\cos 4x}{x^2} = \frac{2\sin^2 2x}{x^2} = 8$$

$$f(0+0) = \lim_{x\to 0} \frac{\sqrt{x}}{(\sqrt{16+\sqrt{x}}-4)} \times \frac{\sqrt{16+\sqrt{x}}+4}{\sqrt{16+\sqrt{x}}+4}$$

$$= \lim_{x\to 0} \frac{\sqrt{x}(\sqrt{16+\sqrt{x}}+4)}{16+\sqrt{x}-16} = 8$$

$\because$ f (0 + 0) = f(0 – 0)

$\therefore$ f(x) can be continuous at x = 0, if

f (0) = a = 8. **Ans.[B]**

16. As given f (0 – 0) = f(0 + 0) = k

Now $f(0-0) = \lim_{h\to 0} \dfrac{\cos\dfrac{(-h)}{2[-h]}}{[-h]}$

$$= \lim_{h\to 0} \frac{\cos\left(\dfrac{-h}{2(-1)}\right)}{-1} = -1$$

$f(0+0) = \lim_{h\to 0} \dfrac{\sin[h]}{[h]+1} = \lim_{h\to 0} \dfrac{\sin 0}{0+1} = 0$

$\because$ f(0 – 0) ≠ f(0 + 0), so k is indeterminate. **Ans.[D]**

17. $f(0-0) = \lim_{h\to 0} (1+|\sin(-h)|)^{a/|\sin(-h)|}$

$= (1+\sin h)^{a/\sin h} = e^a$

$f(0+0) = \lim_{h\to 0} e^{\frac{\tan 2h}{\tan 3h}} = e^{\lim_{h\to 0}\left(\frac{\tan 2h}{\tan 3h}\right)}$

$= e^{\lim_{h\to 0}\frac{2\sec^2 2h}{3\sec^2 3h}} = e^{2/3}$

Now f(x) is continuous at x = 0

$\Rightarrow$ f(0-0) = f(0+0) = f(0)

$\Rightarrow e^a = e^{2/3} = b$

$\therefore$ a = 2/3, b= $e^{2/3}$ **Ans.[A]**

18. $f(x) = |x-1| + |x-2| + |x-3| \Rightarrow f(x) = \begin{cases} 6-3x & x<1 \\ 4-x & 1\le x<2 \\ x & 2\le x<3 \\ 3x-6 & x\ge 3 \end{cases}$

It can be easily varified that f(x) is cont. everywhere, now

$$f'(x) = \begin{cases} -3, & x<1 \\ -1, & 1<x<2 \\ 1, & 2<x<3 \\ 3, & x>3 \end{cases}$$

Hence f(x) is not diff. at x = 1, 2 & 3 **Ans.[B]**

19. In neighborhood of x = 1, $\sin\left(\dfrac{1}{x-1}\right)$ will not have a limiting value but it will be a finite number lying in [-1, 1]. Further $\lim_{x\to 1}(\ln x) = 0$, hence $\lim_{x\to 1}(\ln x)^k = 0$ if k > 0.

$$\therefore \lim_{x\to 1}(\ln x)^k \sin\left(\frac{1}{x-1}\right) = 0 = f(1) \text{ for } k > 0.$$

$$\text{Now } f'(1) = \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{(\ln x)^k \sin\left(\frac{1}{x-1}\right)}{x-1}$$

$$= \lim_{x\to 1}\frac{\ln(1+y)}{y} \times \lim_{x\to 1}(\ln(1+y))^{k-1} \sin\left(\frac{1}{y}\right).$$

By the same argument used above, k must be greater than 1 for $\lim_{x\to 1}(\ln(1+y))^{k-1} \sin\left(\frac{1}{y}\right)$.

Hence f(x) will be diff. if k > 1. **Ans.[B]**

20. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(ax^2 + b\cos \pi x) = a - b$ &

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^4 = 1 = f(1)$

For f(x) to be cont. at x = 1, $a - b = 1$.

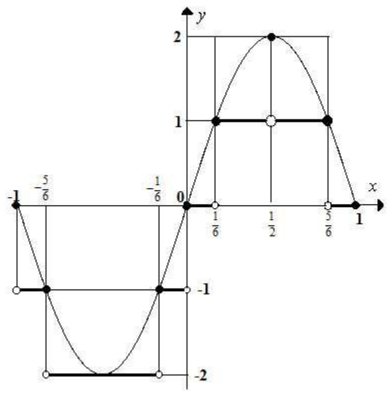
Now further

$$f'(x) = \begin{cases} 2ax - \pi b \sin \pi x, & x < 1 \\ 4x^3, & x > 1 \end{cases}$$

$LHD_{x=1} = 2a$ & $RHD_{x=1} = 4$

For f(x) to be diff. at x = 1, 2a = 4 i.e. a = 2 & b = 1. **Ans.[B]**

21. Refer the adjoining graph.



Ans.[C]

22. $$f(x) = \sin^{-1}\frac{2x}{1+x^2} = \begin{cases} -\pi - 2\tan^{-1} x, & x < -1 \\ 2\tan^{-1} x, & -1 \le x \le 1 \\ \pi - 2\tan^{-1} x, & x > 1 \end{cases}$$

Hence f(x) is cont. everywhere and not diff. at $x = \pm 1$. **Ans.[B]**

23. at x = 0:

$f'(0–0) = \lim_{h\to 0}\frac{|0-h|-0}{-h} = -1$

$f'(0+0) = \lim_{h\to 0}\frac{|0+h|-0}{h} = 1$

Now, since f ‘ (0–0) ≠ f ‘ (0+0)

⇒ f(x) is not differentiable at x = 0. **Ans.[B]**

24. Differentiability at x = 0

$$R\,[f'(0)] = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{(0+h)^2-0}{h} = \lim_{h\to 0} h = 0$$

$$L\,[f'(0)] = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{-(0-h)-0}{-h} = -1$$

∵ R [f ‘ (0)] ≠ L [f ‘(0)]

∴ f(x) is not differentiable at x = 0

Differentiability at x = 1

$$R\,[f'(1)] = \lim_{h\to 0}\frac{f(1+h)^3-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{(1+h)^3-(1+h)+1-1}{h}$$

$$= \lim_{h\to 0}\frac{2h+3h^2+h^3}{h} = 2$$

$$L\,[f'(1)] = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0}\frac{(1-h)-1}{-h}$$

$$= \lim_{h\to 0}\frac{-2h+h^2}{-h} = 2$$

Thus R [f ‘ (1)] = L f ‘(1)]

∴ function f(x) is differentiable at x = 1 **Ans.[B]**

25. Since $f(1-0) = \lim_{x\to 1} 3^x = 3$

$f(1+0) = \lim_{x\to 1} (4-x) = 3$

and $f(1) = 3^1 = 3$

f(1–0) = f(1+0) = f(1)

∴ f(x) is continuous at x = 1

$$\Rightarrow \text{Again } f'(1+0) = \lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x\to 1}\frac{3^x-3}{x-1}$$

$$= \lim_{h\to 0}\frac{3^{1+h}-3}{h}$$

$$= \lim_{h\to 0}\frac{3^{1+h}-3}{h}$$

$$= 3\lim_{h\to 0}\frac{3^h-1}{h}$$

$$= 3\log 3$$

and $f'(1+0) \quad \lim_{x \to 1^-} \frac{f(x)-f(1)}{x-1}$

$$= \lim_{x \to 1} \frac{4-x-3}{x-1} = -1$$

$\therefore \quad f'(1+0) \neq f'(1-0)$

$\neq$ f(x) is not differentiable at x = 1. **Ans.[A]**

26. When x < 0, $f(x) = \frac{x}{1-x}$

$f'(x) = \frac{1}{(1-x)^2}$ ...(1)

which exists finitely for all x < 0

Also when x > 0, $f(x) = \frac{1}{1+x}$

$\Rightarrow f'(x) = \frac{1}{(1+x)^2}$ ...(2)

which exists finitely for all x > 0. Also from (1) and (2) we have

$\begin{cases} f'(0-0)=1 \\ f'(0+0)=1 \end{cases} \Rightarrow f'(0) = 1$

Hence f(x) is differentiable □ $x \in R$ **Ans.[A]**

27. When $x \neq 0$

$f'(x) = 2x \sin\frac{1}{x} + x^2 \cos\frac{1}{x} \cdot \left(-\frac{1}{x^2}\right)$

$= 2x \sin\frac{1}{x} - \cos\left(\frac{1}{x}\right)$

which exists finitely for all $x \neq 0$

and $f'(0) = \lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0} \frac{x^2 \sin 1/x}{x} = 0$

$\therefore$ f is also derivable at x = 0. Thus

$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, x \neq 0 \\ 0, \quad x = 0 \end{cases}$

Also $\lim_{x \to 0} f'(x) = \lim_{x \to 0} \left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$

$= 2 - \lim_{x \to 0} \cos\frac{1}{x}$

But $\lim_{x \to 0} \cos\frac{1}{x}$ does not exist, so $\lim_{x \to 0} f'(x)$ does not exist. Hence f ' is not continuous (so not derivable) at x = 0. **Ans.[B]**

28. $\sqrt{1-x} + \sqrt{1-y} = 1 \Rightarrow y = x + 2\sqrt{1-x} - 1$

$\Rightarrow \frac{dy}{dx} = 1 - \frac{1}{\sqrt{1-x}} = \frac{1-\sqrt{1-x}}{\sqrt{1-x}}$ **Ans.[B]**

29. $x=\ln\left(y+\sqrt{1+y^2}\right)\Rightarrow\sqrt{1+y^2}+y=e^x\ \&\ \sqrt{1+y^2}-y=e^{-x}$

$\Rightarrow y=\dfrac{e^x-e^{-x}}{2}$

$\Rightarrow\dfrac{dy}{dx}=\dfrac{e^x+e^{-x}}{2}=\dfrac{e^{2x}+1}{2e^x}$ **Ans.[C]**

30. for $f(x),\ y=x+\ln x$

then for $f^{-1}(x),\ x=y+\ln y$

$\Rightarrow\dfrac{dx}{dy}=1+\dfrac{1}{y}$ or $\dfrac{dy}{dx}=\dfrac{y}{1+y}$

Further $x+\ln x=1\Rightarrow x=1$, hence for $f^{-1}(x),\ y=1$

$\Rightarrow\dfrac{dy}{dx}=\dfrac{1}{1+1}=\dfrac{1}{2}.$ **Ans.[B]**

31. for $x>\dfrac{1}{2},\ \sin^{-1}\left(3x-4x^3\right)=\pi-3\sin^{-1}x$

Now $y=\pi-3\sin^{-1}x\Rightarrow\dfrac{dy}{dx}=-\dfrac{3}{\sqrt{1-x^2}}$ **Ans.[B]**

32. $x=a\sec\theta\Rightarrow\dfrac{dx}{d\theta}=a\sec\theta\tan\theta,\ y=b\tan\theta\Rightarrow\dfrac{dy}{d\theta}=b\sec^2\theta$

$\Rightarrow\dfrac{dy}{dx}=\dfrac{b\sec^2\theta}{a\sec\theta\tan\theta}=\dfrac{b^2x}{a^2y}$ **Ans.[A]**

33. $\dfrac{dx}{d\theta}=a\ (1+\cos),\ \dfrac{dx}{d\theta}=a\sin\theta$

$\therefore\ \dfrac{dy}{dx}=\dfrac{dy/d\theta}{dx/d\theta}=\dfrac{a\sin\theta}{a(1+\cos\theta)}=\tan\dfrac{1}{2}\theta$ **Ans.[C]**

34. $y=\log e^x-\log(e^x+1)$
$=x-\log(e^x+1)$

$\therefore\ \dfrac{dy}{dx}=1-\dfrac{e^x}{e^x+1}=\dfrac{1}{e^x+1}$ **Ans.[A]**

35. $\dfrac{dy}{dx}=\dfrac{-2x}{(x^2-a^2)^2}\Rightarrow\dfrac{d^2y}{dx^2}$

$=-\dfrac{(x^2-a^2)^2.2-2x.2(x^2-a^2).2x}{(x^2-a^2)^4}$

$=\dfrac{2(3x^2+a^2)}{(x^2-a^2)^3}$ **Ans.[C]**

36. $y = \frac{\sec x - \tan x}{\sec x + \tan x} \cdot \frac{\sec x - \tan x}{\sec x - \tan x}$

$= (\sec x - \tan x)^2/1$

$\therefore \frac{dy}{dx} = 2(\sec x - \tan x)(\sec x \tan x - \sec^2 x)$

$= -2 \sec x (\sec x - \tan x)^2$ **Ans.[B]**

37. Let us first express y in terms of x because all alternatives are in terms of x. So

$x\sqrt{1+y} = -y\sqrt{1+x}$

$\Rightarrow x^{2} (1+y) = y^2 (1+x)$

$\Rightarrow x^2 - y^2 + x^2y - y^2 x = 0$

$\Rightarrow (x-y)(x+y+xy) = 0$

$\Rightarrow x + y + xy = 0$ $(\because x \neq y)$

$\Rightarrow y = -\frac{x}{1-x}$

$\therefore \frac{dy}{dx} = -\frac{(1+x)1 - x.1}{(1+x)^2} = -\frac{1}{(1+x)^2}$ **Ans.[B]**

38. Taking log on both sides, we have

$y \log x + x \log y = 0$

Now using partial derivatives, we have

$\frac{dy}{dx} = -\frac{y/x + \log y}{\log x + x/y} = -\frac{y(y + x\log y)}{x(x + y\log x)}$ **Ans [D]**

39. Here $y = \sqrt{\sin x + y}$ $\Rightarrow y^2 = \sin x + y$

$\therefore 2y\frac{dy}{dx} = \cos x + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y-1}$ **Ans.[B]**

40. $y = e^{x+y}$

$\Rightarrow \log y = x + y \Rightarrow \frac{1}{y}\frac{dy}{dx} = 1 + \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{1-y}$ **Ans.[C]**