CONTINUITY & DIFFERENTIABILITY

EXERCISE 1(A)

1. (d)

L.H.L. at x = 3, $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x+\lambda) = \lim_{h\to 0}(3-h+\lambda) = 3+\lambda$ …..(i)

R.H.L. at x = 3, $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(3x-5) = \lim_{h\to 0}\{3(3+h)-5\} = 4$ …..(ii)

Value of function $f(3) = 4$ …..(iii)

For continuity at x = 3

Limit of function = value of function 3 + λ = 4 $\Rightarrow$ λ = 1.

2. (c)

If function is continuous at x = 0, then by the definition of continuity $f(0) = \lim_{x\to 0} f(x)$

Since f(0) = k. Hence, $f(0) = k = \lim_{x\to 0}(x)\left(\sin\frac{1}{x}\right)$

$\Rightarrow$ k = 0 (a finite quantity lies between –1 to 1)

$\Rightarrow$ k = 0.

3. (c)

Since f(x) is continuous at x = 1,

$\Rightarrow \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$ …..(i)

Now $\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} 2(1-h)+1 = 3$ *i.e.*, $\lim_{x\to 1^-} f(x) = 3$

Similarly, $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} 5(1+h)-2$ *i.e.*, $\lim_{x\to 1^+} f(x) = 3$

So according to equation (i), we have k = 3.

4. (d)

We have $\lim_{x\to 0} f(x) = \lim_{x\to 0}\sin\frac{1}{x}$ = An oscillating number which oscillates between –1 and 1.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Consequently f(x) cannot be continuous at x = 0 for any value of k.

5. (c)

LHL = $\lim_{x\to 1^-} f(x) = \lim_{h\to 0} m(1-h)^2 = m$

RHL = $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} 2(1+h) = 2$ and $f(1) = m$

Function is continuous at x = 1, $\therefore$ LHL = RHL = f(1)

Therefore m = 2.

6. (a)

$\lim_{x\to 0}(\cos x)^{1/x} = k \Rightarrow \lim_{x\to 0}\frac{1}{x}\log(\cos x) = \log k$

$\Rightarrow \lim_{x\to 0}\frac{1}{x}\lim_{x\to 0}\log\cos x = \log k$

$\Rightarrow \lim_{x\to 0}\frac{1}{x}\times 0 = \log_e k \Rightarrow k = 1$.

7. (b)

Since f is continuous at $x=\frac{\pi}{4}$; $\therefore f\left(\frac{\pi}{4}\right)=\underset{h\to 0}{f}\left(\frac{\pi}{4}+h\right)=\underset{h\to 0}{f}\left(\frac{\pi}{4}-h\right)$

$\Rightarrow \frac{\pi}{4}+b=\frac{\pi}{4}+a^2 \Rightarrow b=a^2$

Also as f is continuous at $x=\frac{\pi}{2}$;

$\therefore f\left(\frac{\pi}{2}\right)=\lim_{h\to 0} f\left(\frac{\pi}{2}+h\right)=\lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)$

$\Rightarrow 2b+a=b \Rightarrow a=-b$.

Hence (–1, 1) & (0, 0) satisfy the above relations.

8. (c)

$\lim_{x\to 1^-} f(x)=\lim_{h\to 0} f(1-h) = \lim_{h\to 0}\left[2+\sin\frac{\pi}{2}(1-h)\right] = 3$

Similarly, $\lim_{x\to 1^+} f(x)=\lim_{h\to 0} f(1+h) = \lim_{h\to 0} a(1+h)+b = a + b$

$\because$ f(x) is continuous at x = 1 so $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=f(1)$

$\Rightarrow a+b=3$ …….(i)

Again, $\lim_{x\to 2^-} f(x)=\lim_{h\to 0} f(2-h)=\lim_{h\to 0} a(2-h)+b=2a+b$

and $\lim_{x\to 2^+} f(x)=\lim_{h\to 0} f(2+h) = \lim_{h\to 0} \tan\frac{\pi}{8}(2+h) = 1$

f(x) is continuous in $(-\infty, 6)$, so it is continuous at $x=2$ also, so

$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$

$\Rightarrow 2a+b=1$ ……(ii)

Solving (i) and (ii) a = - 2, b = 5.

9. (a)

$\lim_{x\to \frac{\pi}{2}^-} f(x)=\frac{\pi}{2}, \lim_{x\to \frac{\pi}{2}^+} f(x)=-\frac{\pi}{2}$

Since $\lim_{x\to \frac{\pi}{2}^-} f(x)\neq \lim_{x\to \frac{\pi}{2}^+} f(x)$,

$\therefore$ Function is discontinuous at $x=\frac{\pi}{2}$

10. (b)

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\left(\frac{2\sin^2 3x}{(3x)^2}\right)3=6$ and

$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{9+\sqrt{x}}-3}=\lim_{x\to 0^+}\left(\sqrt{9+\sqrt{x}}+3\right)=6$

Hence $a=6$.

11. (c)

The function $f(x)=\frac{1}{x^2+x-6}$ is discontinuous at 2 points.

The function $f(x)=\frac{1}{x^2+x-6}$ & $g(x)=\frac{1}{x-1} \Rightarrow g(f(x))=\frac{1}{x^2+x-7}$

g(f(x)) is discontinuous at 4 points.

Hence, the composite $f(g(x))$ is discontinuous at three points $x=\frac{2}{3}, 1$ & $\frac{3}{2}$

12. (b)

$$\lim_{x\to 0}\frac{\ln b \ln(a+x)-\ln a \ln(b-x)}{x}=\lim_{x\to 0}\frac{\ln b\left(\ln(a+x)-\ln a\right)-\ln a\left(\ln a \ln(b-x)-\ln b\right)}{x}$$

$$=\ln b \lim_{x\to 0}\frac{\left(\ln(a+x)-\ln a\right)}{x}+\ln a \lim_{x\to 0}\frac{\left(\ln(b-x)-\ln b\right)}{x}$$

$$=\frac{\ln b}{a}\lim_{x\to 0}\frac{\ln\left(1+\frac{x}{a}\right)}{\frac{x}{a}}+\frac{\ln a}{b}\lim_{x\to 0}\frac{\ln\left(1+\frac{x}{b}\right)}{\frac{x}{b}}$$

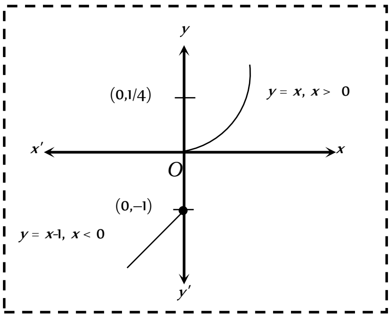
$$=\frac{\ln b}{a}+\frac{\ln a}{b}=\frac{\ln\left(b^b a^a\right)}{ab}$$

13. (b)

$$f(2)=2,\ f(2^+)=\lim_{x\to 2^+}\frac{x^2-5x+6}{x^2-4}=\lim_{x\to 2^+}\frac{(x-3)}{(x+2)}=-\frac{1}{4}$$

14. (c)

Clearly from curve drawn of the given function $f(x)$, it is discontinuous at $x=0$.



15. (b)

$$f(x)=\begin{cases}(1+|\tan x|)^{\frac{a}{3|\tan x|}}, & -\frac{\pi}{6}<x<0\\ b, & x=0\\ e^{\frac{\tan 6x}{\tan 3x}}, & 0<x<\frac{\pi}{6}\end{cases}$$

For f(x) to be continuous at x = 0

$\Rightarrow \lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$

$\Rightarrow \lim_{x\to 0}(1+|\tan x|)^{\frac{a}{3|\tan x|}} = e^{\lim_{x\to 0^-}\left((1+|\tan x|-1)\frac{a}{3|\tan x|}\right)} = e^{a/3}$

Now, $\lim_{x\to 0^+} e^{\frac{\tan 6x}{\tan 3x}} = \lim_{x\to 0^+} e^{\left(\frac{\tan 6x}{6x}.6x\right)\Big/\left(\frac{\tan 3x}{3x}.3x\right)} = e^2$

$\therefore\ e^{a/3} = b = e^2 \Rightarrow a = 6$ and $b = e^2$.

16. (d)

Let $f(x) = \ln\frac{x}{4}$

$\lim_{x\to 4} xf(x) = \lim_{x\to 4} x\ln\frac{x}{4} = 0$

17. (a)

Note that $[x+2] = 0$ if $0 \le x+2 < 1$

i.e. $[x+2] = 0$ if $-2 \le x < -1$.

Thus domain of f is $R-[-2,-1)$

We have $\sin\left(\frac{\pi}{[x+2]}\right)$ is continuous at all points of $R-[-2,-1)$ and $[x]$ is continuous on $R-I$, where I denotes the set of integers.

Thus the points where f can possibly be discontinuous are……, $-3,-2,-1,0\,1,2,.........$. But for $-1 \le x < 0, [x+1] = 0$ and $\sin\left(\frac{\pi}{[x+2]}\right)$ is defined.

Therefore f(x) = 0 for $-1 \le x < 0$.

Also $f(x)$ is not defined on $-2 \le x < -1$.

Hence set of points of discontinuities of f(x) is $I-\{-1\}$.

18. (b)

$f(x) = \lim_{x\to 0}\left(\frac{2x-\sin^{-1}x}{2x+\tan^{-1}x}\right) = f(0) \quad ,\left(\frac{0}{0}\text{form}\right)$

Applying L-Hospital's rule, $f(0) = \lim_{x\to 0}\frac{\left(2-\frac{1}{\sqrt{1-x^2}}\right)}{\left(2+\frac{1}{1+x^2}\right)} = \frac{2-1}{2+1} = \frac{1}{3}$

19. (d)

For continuity at all $x \in R$, we must have

$f\left(-\frac{\pi}{2}\right) = \lim_{x\to(-\pi/2)^-}(4\sin x) = \lim_{x\to(-\pi/2)^+}(a\sin x - b)$

$\Rightarrow\ 4 = -a-b$ …..(i)

and $f\left(\frac{\pi}{2}\right) = \lim_{x\to(\pi/2)^-}(a\sin x - b) = a - b = \lim_{x\to(\pi/2)^+}(\cos x) = 0$

$\Rightarrow\ 0 = a - b$ ….(ii)

From (i) and (ii), $a = -2$ and $b = -2$.

20. (a)

$$f(5) = \lim_{x\to 5} f(x) = \lim_{x\to 5}\frac{x^2 - 10x + 25}{x^2 - 7x + 10} = \lim_{x\to 5}\frac{(x-5)^2}{(x-2)(x-5)} = \frac{5-5}{5-2} = 0 .$$

21. (c)

For continuity at 0, we must have $f(0) = \lim_{x\to 0} f(x)$

$$= \lim_{x\to 0}(x+1)^{\cot x} = \lim_{x\to 0}\left\{(1+x)^{\frac{1}{x}}\right\}^{x\cot x} = \lim_{x\to 0}\left\{(1+x)^{\frac{1}{x}}\right\}^{\lim_{x\to 0}\left(\frac{x}{\tan x}\right)} = e.$$

22. (a)

Conceptual question

23. (c)

f(x) is continuous at $x = \frac{\pi}{3}$, then $\lim_{x\to\pi/3} f(x) = f(0)$ or

$$\lambda = \lim_{x\to\pi/3}\frac{1-\sin\frac{3x}{2}}{\pi - 3x},\left(\frac{0}{0}\text{ form}\right)$$

Applying L-Hospital's rule, $\lambda = \lim_{x\to\pi/3}\frac{-\frac{3}{2}\cos\frac{3x}{2}}{-3} = 0$

24. (d)

If f(x) is continuous at x = 0 then,

$$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{2-\sqrt{x+4}}{\sin 2x},\left(\frac{0}{0}\text{ form}\right)$$

Using L–Hospital's rule, $f(0) = \lim_{x\to 0}\frac{\left(-\frac{1}{2\sqrt{x+4}}\right)}{2\cos 2x} = -\frac{1}{8}$.

25. (d)

$x^2 + 2 = 3x \Rightarrow x = 1, 2$

F(x) will be continuous only at x = 1 & 2.

26. (b)

$$f(x) = \left[x^2 + e^{\frac{1}{2-x}}\right]^{-1} \text{ and } f(2) = k$$

If f(x) is continuous from right at x = 2 then $\lim_{x\to 2^+} f(x) = f(2) = k$

$$\Rightarrow \lim_{x\to 2^+}\left[x^2 + e^{\frac{1}{2-x}}\right]^{-1} = k \Rightarrow k = \lim_{h\to 0} f(2+h) \Rightarrow k = \lim_{h\to 0}\left[(2+h)^2 + e^{\frac{1}{2-(2+h)}}\right]^{-1}$$

$$\Rightarrow k = \lim_{h\to 0}\left[4 + h^2 + 4h + e^{-1/h}\right]^{-1} \Rightarrow k = [4+0+0+e^{-\infty}]^{-1} \Rightarrow k = \frac{1}{4}.$$

27. (c)

$$\lim_{x\to\pi} f(x) = \lim_{x\to\pi}\frac{2\cos^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}} = \lim_{x\to\pi}\frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}} = \lim_{x\to\pi}\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)$$

$\therefore$ At $x = \pi,\ f(\pi) = -\tan\frac{\pi}{4} = -1$.

28. (c)

$$\text{L.H.L.} = \lim_{x\to 0^{-1}}\frac{\sqrt{4+kx} - \sqrt{4-kx}}{x} = \lim_{x\to 0^{-1}}\frac{2kx}{x}\times\frac{1}{\sqrt{4+kx}+\sqrt{4-kx}} = \frac{k}{2}$$

$$\text{R.H.L.} = \lim_{x\to 0^+}\frac{2x^2+3x}{\sin x} = \lim_{x\to 0^+}\frac{x}{\sin x}(2x+3) = 3$$

Since it is continuous, hence L.H.L = R.H.L $\Rightarrow k = 6$.

29. (c)

$|x|$ is continuous at $x = 0$ and $\frac{|x|}{x}$ is discontinuous at $x = 0$

$\therefore f(x) = |x| + \frac{|x|}{x}$ is discontinuous at $x = 0$.

30. (b)

$$\lim_{x\to 0^+}\frac{x\left(e^x - 1\right)}{|\tan x|} = \lim_{x\to 0^+}\frac{x\left(e^x-1\right)}{\tan x} = 0$$

$$\lim_{x\to 0^-}\frac{x\left(e^x - 1\right)}{|\tan x|} = -\lim_{x\to 0^-}\frac{x\left(e^x-1\right)}{\tan x} = 0$$

So f(x) is continuous at x = 0.

$$\text{Now L.H.D.} = \lim_{x\to 0^-}\frac{\frac{x\left(e^x-1\right)}{|\tan x|} - 0}{x - 0} = -\lim_{x\to 0^-}\frac{x}{\tan x}\times\frac{e^x-1}{x} = -1$$

$$\text{R.H.D.} = \lim_{x\to 0^+}\frac{\frac{x\left(e^x-1\right)}{|\tan x|} - 0}{x - 0} = \lim_{x\to 0^-}\frac{x}{\tan x}\times\frac{e^x-1}{x} = 1$$

$\text{L.H.D.} \neq \text{R.H.D.}$

F(x) is continuous but not differentiable at $x = 0$

31. (a)

We have, $f(x)=\dfrac{x}{1+|x|}=\begin{cases}\dfrac{x}{1+x} & , \quad x>0 \\ 0 & , \quad x=0 \\ \dfrac{x}{1-x} & , \quad x<0\end{cases}$;

$$\text{L.H.D.}=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}=\lim_{h\to 0}\frac{\frac{-h}{1+h}-0}{-h}=1$$

$$\text{R.H.D.}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\frac{\lim_{h\to 0}\frac{h}{1+h}-0}{h}=\lim_{h\to 0}\frac{1}{1+h}=1$$

So, f(x) is differentiable at x = 0; Also f(x) is differentiable at all other points.

Hence, f(x) is everywhere differentiable.

32. (b)

Let $f(x)=|x-1|+|x-3|=\begin{cases}-(x-1)-(x-3) & , \quad x<1 \\ (x-1)-(x-3) & , \quad 1\le x<3 \\ (x-1)+(x-3) & , \quad x\ge 3\end{cases}=\begin{cases}-2x+4 & , \quad x<1 \\ 2 & , \quad 1\le x<3 \\ 2x-4 & , \quad x\ge 3\end{cases}$

Since, $f(x)=2$ for $1\le x<3$. Therefore $f'(x)=0$ for all $x\in(1,3)$.

Hence, $f'(x)=0$ at $x=2$.

33. (a)

We have, $\lim_{x\to 0}f(x)=\lim_{x\to 0}\frac{\sin x^2}{x}=\lim_{x\to 0}\left(\frac{\sin x^2}{x^2}\right)x=1\times 0=0=f(0)$

So, f(x) is continuous at x = 0,

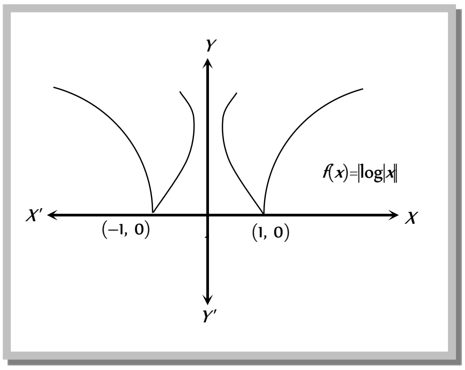
$f(x)$ is also derivable at

x = 0, because $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{\sin x^2}{x^2}=1$

exists finitely.

34. (b)

It is evident from the graph of $f(x)=|\log|x||$ than $f(x)$ is everywhere continuous but not differentiable at $x=\pm 1$.



35. (a)

$f(x)=[x]\sin(\pi x)$

If x is just less than k, [x] = k – 1. $\therefore$ $f(x)=(k-1)\sin(\pi x)$, when $x<k \ \forall \ k\in I$

Now L.H.D. at $x=k$,

$$=\lim_{x\to k}\frac{(k-1)\sin(\pi x)-k\sin(\pi k)}{x-k}=\lim_{x\to k}\frac{(k-1)\sin(\pi x)}{(x-k)} \quad [\text{as } \sin(\pi k)=0 \quad k\in I]$$

$= \lim_{h\to 0}\dfrac{(k-1)\sin(\pi(k-h))}{-h}$ [Let $x=(k-h)$]

$= \lim_{h\to 0}\dfrac{(k-1)(-1)^{k-1}\sin h\pi}{-h}$

$= \lim_{h\to 0}(k-1)(-1)^{k-1}\dfrac{\sin h\pi}{h\pi}\times(-\pi)$

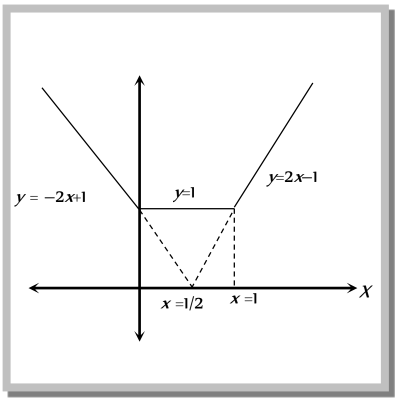
$= (k-1)(-1)^k\pi = (-1)^k(k-1)\pi$.

36. (a)

We have, $f(x)=|x|+|x-1|=\begin{cases}-2x+1, & x<0\\ 1, & 0\le x<1\\ 2x-1, & x\ge 1\end{cases}$

Since, $\lim_{x\to 1^-}f(x)=\lim_{x\to 1^-}1=1, \lim_{x\to 1^+}f(x)=\lim_{x\to 1^+}(2x-1)=1$ and $f(1)=2\times 1-1=1$

$\therefore \lim_{x\to 1^-}f(x)=\lim_{x\to 1^+}f(x)=f(1)$. So, f(x) is continuous at *x* = 1.



Now, $\lim_{x\to 1^-}\dfrac{f(x)-f(1)}{x-1}=\lim_{h\to 0}\dfrac{f(1-h)-f(1)}{-h}=\lim_{h\to 0}\dfrac{1-1}{-h}=0$ and

$\lim_{x\to 1^+}\dfrac{f(x)-f(1)}{x-1}=\lim_{h\to 0}\dfrac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\dfrac{2(1+h)-1-1}{h}=2$.

$\therefore$ (LHD at x = 1) $\neq$ (RHD atx = 1).

So, *f*(*x*) is not differentiable at *x* = 1.

Alternately

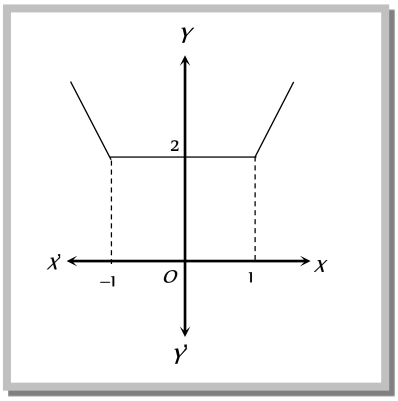
By graph, it is clear that the function is not differentiable at x = 0, 1 as there it has sharp edges.

37. (c)

Here $f(x)=|x-1|+|x+1| \Rightarrow f(x)=\begin{cases}2x, & \text{when } x>1\\ 2, & \text{when } -1\le x\le 1\\ -2x, & \text{when } x<-1\end{cases}$

Alternate

The graph of the function is shown alongside,



From the graph it is clear that the function is continuous at all real *x*, also differentiable at all real *x* except at $x=\pm 1;$ Since sharp edges at $x=-1$ and $x=1$.

At $x=1$ we see that the slope from the right *i.e.*, R.H.D. = 2, while slope from the left *i.e.*, L.H.D.= 0

Similarly, at $x=-1$ it is clear that R.H.D. = 0 while L.H.D. = – 2

Here, $f'(x)=\begin{cases}-2, & x<-1\\ 0, & -1<x<1\\ 2, & x>1\end{cases}$ (No equality on –1 and +1)

Now, at $x=1$, $f'(1^+)=2$ while $f'(1^-)=0$ and

at $x=-1$, $f'(-1^+)=0$ while $f'(-1^-)=-2$

Thus, f(x) is not differentiable at $x=\pm 1$.

38. (d)

$\lim_{x\to -1^-} f(x)=\lim_{x\to -1^{-1}}\left(ax^2+bx+2\right)=a-b+2$ and

$\lim_{x\to -1^+} f(x)=\lim_{x\to -1^+}\left(bx^2+ax+4\right)=b-a+4$

For continuity $a-b+2=b-a+4 \Rightarrow a-b=1...(i)$

Now $f'(x)=\begin{cases}2ax+b & , \quad x<-1\\ 2bx+a & , \quad x>-1\end{cases} \Rightarrow \text{R.H.D.}=-2a+b \;\&\; \text{L.H.D.}=-2b+a$

For differentiability $-2a+b=-2b+a \Rightarrow a=b...(ii)$

From (i) & (ii) no value of (a, b) is possible.

39. (b)

$h(x)=e^{(f(x))^3+(g(x))^3+x} \Rightarrow h'(x)=e^{(f(x))^3+(g(x))^3+x}\left(3(f(x))^2 f'(x)+3(g(x))^2 g'(x)+1\right)$

$\Rightarrow h'(x)=h(x)\left(3(f(x))^2\frac{g(x)}{f(x)}-3(g(x))^2\frac{f(x)}{g(x)}+1\right)$

$\Rightarrow h'(x)=h(x) \Rightarrow h(x)=e^{x+c}$

Now $h(5)=e^6 \Rightarrow h(x)=e^{x+1}$

Hence $h(10)=e^{11}$

40. (c)

$[2+h]=2, [2-h]=1, [1+h]=1, [1-h]=0$

At *x* = 2, we will check $\text{RHL}=\text{LHL}=f(2)$

$\text{RHL}=\lim_{h\to 0}|4+2h-3|[2+h]=2, f(2)=1.2=2$

$\text{LHL}=\lim_{h\to 0}|4-2h-3|[2-h]=1, R\neq L$, $\therefore$ not continuous

At $x=1, \text{RHL}=\lim|2+2h-3|[1+h]=1.1=1,$

$f(1)=|-1|[1]=1$

$\text{LHL}=\lim_{h\to 0}\sin\frac{\pi}{2}(1-h)=1$

continuous at x = 1

$\text{R.H.D.}=\lim_{h\to 0}\frac{|2+2h-3|[1+h]-1}{h}=\lim_{h\to 0}\frac{|-1|.1-1}{h}=\lim_{h\to 0}\frac{1-1}{h}=0$

$\text{L.H.D.}=\lim_{h\to 0}\frac{|2-2h-3|[1-h]-1}{-h}=\lim_{h\to 0}\frac{1.0-1}{-h}=\lim_{h\to 0}\frac{1}{h}=\infty$

Since R.H.D. $\neq$ L.H.D. $\therefore$ not differentiable. at x = 1.

41. (b)

Clearly, $f(x)$ is differentiable for all non-zero values of *x*,

For $x \neq 0$, we have $f'(x) = \dfrac{xe^{-x^2}}{\sqrt{1-e^{-x^2}}}$

Now, (L.H.D. at *x* = 0)

$= \lim_{x\to 0^-} \dfrac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \dfrac{f(0-h)-f(0)}{-h}$

$= \lim_{h\to 0} \dfrac{\sqrt{1-e^{-h^2}}}{-h} = \lim_{h\to 0} -\dfrac{\sqrt{1-e^{-h^2}}}{h}$

$= -\lim_{h\to 0} \sqrt{\dfrac{e^{h2}-1}{h^2}} \times \dfrac{1}{\sqrt{e^{h^2}}} = -1$

and, (RHD at *x* = 0) = $\lim_{x\to 0^+} \dfrac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \dfrac{\sqrt{1-e^{-h^2}}-0}{h}$

$= \lim_{h\to 0} \sqrt{\dfrac{e^{h^2}-1}{h^2}} \times \dfrac{1}{\sqrt{e^{h^2}}} = 1$.

So, $f(x)$ is not differentiable at $x = 0$,

Hence, the points of differentiability of $f(x)$ are $(-\infty, 0) \cup (0, \infty)$

42. (a)

We have, $f(x) = \begin{cases} e^{\sin x}, -\dfrac{\pi}{2} \leq x < 0 \\ e^{-\sin x}, 0 \leq x \leq \dfrac{\pi}{2} \end{cases}$

Clearly, $f(x)$ is continuous and differentiable for all non-zero *x*.

Now, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0} e^{\sin x} = 1$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0} e^{-\sin x} = 1$

Also, $f(0) = e^0 = 1$

So, f(x) is continuous for all *x*.

(LHD at x = 0) = $\left(\dfrac{d}{dx}(e^x)\right)_{x=0} = (e^x)_{x=0} = e^0 = 1$

(RHD at $x = 0$) = $\left(\dfrac{d}{dx}(e^{-x})\right)_{x=0} = (-e^{-x})_{x=0} = -1$

So, f(x) is not differentiable at x = 0.

43. (b)

We have, $f(x) = \sqrt{1-\sqrt{1-x^2}}$. The domain of definition of $f(x)$ is [–1, 1].

For $x \neq 0, x \neq 1, \quad x \neq -1$ we have $f'(x) = \dfrac{1}{\sqrt{1-\sqrt{1-x^2}}} \times \dfrac{x}{\sqrt{1-x^2}}$

Since $f(x)$ is not defined on the right side of $x = 1$ and on the left side of $x = -1$.

Also, $f'(x) \to \infty$ when $x \to -1^+$ or $x \to 1^-$.

So, we check the differentiability at $x = 0$.

Now, (LHD at $x=0$) = $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$

$= \lim_{h\to 0}\frac{\sqrt{1-\sqrt{1-h^2}}-0}{-h}=-\lim_{h\to 0}\frac{\sqrt{1-\{1-(1/2)h^2+(3/8)h^4+....\}}}{h}$

$= -\lim_{h\to 0}\sqrt{\frac{1}{2}-\frac{3}{8}h^2+.....}=-\frac{1}{\sqrt{2}}$

Similarly, (RHD at $x = 0$) = $\frac{1}{\sqrt{2}}$

Hence, $f(x)$ is not differentiable at $x = 0$.

44. (d) Since $f(x)$ is differentiable at $x=c$, therefore it is continuous at $x=c$.

Hence, $\lim_{x\to c} f(x)=f(c)$.

45. (c)

$(x^2-3x+2)=(x-1)(x-2)>0$ When $x<1$ or >2,

And $(x^2-3x+2)=(x-1)(x-2)<0$ when $1\le x\le 2$

Also $\cos|x|=\cos x$

$\therefore\ f(x)=-(x^2-4)(x^2-3x+2)+\cos x,\ \ 1\le x\le 2$

and $f(x)=(x^2-4)(x^2-3x+2)+\cos x,\ \ x<1$ or $x>2$

Evidently $f(x)$ is not differentiable at x = 1.

46. (b)

$f(0)=0$ and $f(x)=x^2e^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}$

R.H.L. = $\lim_{h\to 0}(0+h)^2e^{-2/h}=\lim_{h\to 0}\frac{h^2}{e^{2/h}}=0$

L.H.L. = $\lim_{h\to 0}(0-h)^2e^{-\left(\frac{1}{h}-\frac{1}{h}\right)}=0$

$\therefore$ f(x) is continuous at x = 0.

R.H.D. at $(x=0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{h^2e^{-2/h}}{h}=he^{-2/h}=0$

L.H.D. at $(x=0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{h^2e^{-\left(\frac{1}{h}-\frac{1}{h}\right)}}{-h}=\lim_{h\to 0}(-h)=0$

F(x) is differentiable at x = 0

47. (d)

$\lim_{x\to 0}f(x)=x^3\sin^2\left(\frac{1}{x}\right)=0$ as $0\le\sin^2\left(\frac{1}{x}\right)\le 1$ and $x\to 0$

Therefore $f(x)$ is continuous at $x=0$.

Also, the function $f(x)=x^3\sin^2\frac{1}{x}$ is differentiable because

$RHD=\lim_{h\to 0}\frac{h^3\sin^2\frac{1}{h}-0}{h}=\lim_{h\to 0}h^2\sin^2\frac{1}{h}=0$, $LHD=\lim_{h\to 0}\frac{h^3\sin\left(\frac{1}{-h}\right)}{-h}=0$.

48. (b)

49. (d)

50. (c)

$\lim_{h\to 0^-} 1+(2-h)=3$, $\lim_{h\to 0^+} 5-(2+h)=3$, $f(2)=3$

Hence, *f* is continuous at $x=2$

Now $RHD=\lim_{h\to 0}\frac{5-(2+h)-3}{h}=-1$

$LHD=\lim_{h\to 0}\frac{1+(2-h)-3}{-h}=1$

$\therefore$ f(x) is not differentiable at $x=2$.

51. (c)

$g(x)=|f(|x|)|\geq 0$. So $g(x)$ cannot be onto.

If $f(x)$ is one-one and $f(x_1)=-f(x_2)$ then $g(x_1)=g(x_2)$.

So, ' $f(x)$ is one-one' does not ensure that $g(x)$ is one-one.

If f(x) is continuous for $x\in R$, $|f(|x|)|$ is also continuous for $x\in R$.

So the answer (c) is correct.

The fourth answer (d) is not correct as f(x) being differentiable does not ensure |f(x)| being differentiable.

52. (b)

Given $f(4)=6, f'(4)=1$

$\therefore \lim_{x\to 4}\frac{xf(4)-4f(x)}{x-4}=\lim_{x\to 4}\frac{xf(4)-4f(4)+4f(4)-4f(x)}{x-4}$

$=\lim_{x\to 4}\frac{(x-4)f(4)}{x-4}-4\lim_{x\to 2}\frac{f(x)-f(4)}{x-4}$

$= f(4)-2f'(4)=4$

53. (c)

$f(x+2y)=2f(x)f(y)\Rightarrow 2f'(x+2y)=2f(x)f'(y)$ {partially differentiating w.r.to y}

For x = 5 & y = 0, $f'(5)=f(5)f'(0)\Rightarrow f'(5)=6$

54. (c)

By L'hospital's rule

$\lim_{x\to 2}\frac{g^2(x)f^2(2)-f^2(x)g^2(2)}{x^2-4}=\lim_{x\to 2}\frac{g(x)g'(x)f^2(2)-f(x)f'(x)g^2(2)}{x}$

$=\frac{(-1)\times 4\times 9-3\times(-2)\times 1}{2}=-15$

55. (b)

Given $5f(2x)+3f\left(\frac{2}{x}\right)=2x+2$ ......(i)

Replacing *x* by $\frac{1}{x}$ in (i), $5f\left(\frac{2}{x}\right)+3f(2x)=\frac{2}{x}+2$ ......(ii)

On solving equation (i) and (ii), we get, $8f(2x)=5x-\frac{3}{x}+2$,

$$\Rightarrow 8f(x)=\frac{5x}{2}-\frac{6}{x}+2$$

$$\therefore\ 8f'(x)=\frac{5}{2}+\frac{6}{x^2}$$

$$\because\ y=xf(x)\ \Rightarrow\ \frac{dy}{dx}=f(x)+xf'(x)$$

$$=\frac{1}{8}\left(\frac{5x}{2}-\frac{6}{x}+2\right)+\frac{x}{8}\left(\frac{5}{2}+\frac{6}{x^2}\right)$$

at $x=1$, $\frac{dy}{dx}=\frac{1}{8}\left(\frac{5}{2}-6+2\right)+\frac{1}{8}\left(\frac{5}{2}+6\right)=\frac{7}{8}$

56. (d)

$$f(x)=\begin{cases}x^3-1, & x\ge 1\\ 1-x^3, & x<1\end{cases}\quad \text{and}\quad f'(x)=\begin{cases}3x^2, & x\ge 1\\ -3x^2, & x<1\end{cases}$$

$$f'(1^+)=3,\ f'(1^-)=-3$$

57. (b)

$$f(x)=\sin 2x.\cos 2x.\cos 3x+\log_2 2^{x+3},$$

$$\Rightarrow f(x)=\frac{1}{2}\sin 4x\cos 3x+(x+3)\log_2 2,$$

$$\Rightarrow f(x)=\frac{1}{4}[\sin 7x+\sin x]+x+3$$

Differentiate w.r.t. x,

$$f'(x)=\frac{1}{4}[7\cos 7x+\cos x]+1,$$

$$\Rightarrow f'(\pi)=-2+1=-1.$$

58. (b) In neighborhood of $x=\frac{3\pi}{4}$, $|\cos^3 x|=-\cos^3 x$ and $|\sin^3 x|=\sin^3 x$

$$\therefore\ y=-\cos^3 x+\sin^3 x$$

$$\therefore\ \frac{dy}{dx}=3\cos^2 x\sin x+3\sin^2 x\cos x$$

At $x=\frac{3\pi}{4}$, $\frac{dy}{dx}=3\cos^2\frac{3\pi}{4}\sin\frac{3\pi}{4}+3\sin^2\frac{3\pi}{4}\cos\frac{3\pi}{4}=0$.

59. (b)

$$f(x)=\log_x(\log x)=\frac{\log(\log x)}{\log x}$$

$$\Rightarrow f'(x)=\frac{\frac{1}{x}-\frac{1}{x}\log(\log x)}{(\log x)^2}$$

$$\Rightarrow f'(e)=\frac{\frac{1}{e}-0}{1}=\frac{1}{e}$$

60. (d)

$$f(x)=|\log x|=\begin{cases}-\log x, & \text{if } 0<x<1\\ \log x, & \text{if } x\ge 1\end{cases}$$

$$\Rightarrow f'(x)=\begin{cases}-\frac{1}{x}, & \text{if } 0<x<1\\ \frac{1}{x}, & \text{if } x>1\end{cases}.$$

Clearly $f'(1^-)=-1$ and $f'(1^+)=1$,

$\therefore$ $f'(x)$ does not exist at x = 1

61. (c)

Let $y=\left[\log\left\{e^x\left(\frac{x-1}{x+1}\right)\right\}\right]=\log e^x+\log\left(\frac{x-1}{x+1}\right)$

$$\Rightarrow y=x+[\log(x-1)-\log(x+1)]$$

$$\Rightarrow \frac{dy}{dx}=1+\left[\frac{1}{x-1}-\frac{1}{x+1}\right]=1+\frac{2}{(x^2-1)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{x^2+1}{x^2-1}.$$

62. (a)

$$x=\exp\left\{\tan^{-1}\left(\frac{y-x}{x}\right)\right\}\Rightarrow \log x=\tan^{-1}\left(\frac{y-x}{x}\right)$$

$$\Rightarrow \frac{y-x}{x}=\tan(\log x)\Rightarrow y=x\tan(\log x)+x$$

$$\Rightarrow \frac{dy}{dx}=\tan(\log x)+x\frac{\sec^2(\log x)}{x}+1$$

$$\Rightarrow \frac{dy}{dx}=\tan(\log x)+\sec^2(\log x)+1$$

At x = 1, $\frac{dy}{dx}=2$.

63. (a)

$$y=\sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)+\sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$$

$$=\cos^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)+\sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)=\frac{\pi}{2}$$

$$\Rightarrow \frac{dy}{dx}=0$$

64. (d)

$$\frac{d}{dx}\tan^{-1}\left[\frac{\cos x-\sin x}{\cos x+\sin x}\right]=\frac{d}{dx}\tan^{-1}\left[\tan\left(\frac{\pi}{4}-x\right)\right]=-1 .$$

65. (b)

Let $y=\sin^2\left(\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right)$

Put $x=\cos\theta \Rightarrow \theta=\cos^{-1}x$

$$\Rightarrow y=\sin^2\left(\cot^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)=\sin^2\left(\cot^{-1}\left(\tan\frac{\theta}{2}\right)\right)$$

$$\Rightarrow y=\sin^2\left(\frac{\pi}{2}-\frac{\theta}{2}\right)=\cos^2\frac{\theta}{2}=\frac{1}{2}(1+\cos\theta)=\frac{1}{2}(1+x)$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}$$

66. (a)

Let $\cos\alpha=\frac{5}{13}$. Then $\sin\alpha=\frac{12}{13}$. So, $y=\cos^{-1}\{\cos\alpha.\cos x-\sin\alpha.\sin x\}$

$\therefore y=\cos^{-1}\{\cos(x+\alpha)\}=x+\alpha$ ($\because$ $x+\alpha$ is in the first or the second quadrant)

$$\therefore \frac{dy}{dx}=1 .$$

67. (c)

$$y\left(\frac{\tan^2 2x-\tan^2 x}{1-\tan^2 2x\tan^2 x}\right)\cot 3x=\left(\frac{\tan 2x-\tan x}{1+\tan 2x\tan x}\right)\left(\frac{\tan 2x+\tan x}{1-\tan 2x\tan x}\right)\cot 3x$$

$$\Rightarrow y=\tan x\tan 3x\cot 3x=\tan x$$

$$\Rightarrow \frac{dy}{dx}=\sec^2 x$$

68. (a)

$$f(x)=\cot^{-1}\left(\frac{x^x-x^{-x}}{2}\right)$$

Put $x^x=\tan\theta$, $\therefore$ $y=f(x)=\cot^{-1}\left(\frac{\tan^2\theta-1}{2\tan\theta}\right)$

$$=\cot^{-1}(-\cot 2\theta)=\pi-\cot^{-1}(\cot 2\theta)$$

$$\Rightarrow y=\pi-2\theta=\pi-2\tan^{-1}(x^x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^{2x}}.x^x(1+\log x)$$

$$\Rightarrow f'(1) = -1 .$$

69. (a)

$$y = \frac{(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)}{1-x} = \frac{1-x^{16}}{1-x}$$

$$\therefore \frac{dy}{dx} = \frac{-16x^{15}(1-x)+1-x^{16}}{(1-x)^2}, \quad \therefore \text{At } x=0, \frac{dy}{dx}=1.$$

70. (c)

$$f(x) = \frac{2\sin x.\cos x.\cos 2x.\cos 4x}{2\sin x} = \frac{\sin 8x}{8\sin x}$$

$$\therefore f'(x) = \frac{1}{8}.\frac{8\cos 8x.\sin x - \cos x.\sin 8x}{\sin^2 x}$$

$$\therefore f'\left(\frac{\pi}{4}\right) = 0.$$

71. (a)

$$xe^{x+y} = y + 2\sin x \Rightarrow e^{x+y} + xe^{x+y}(1+y') = y' + 2\cos x$$

Now x = 0 gives y = 0, hence $\frac{dy}{dx} = -1$.

72. (a)

$$\sin(3x-2y) = \log(3x-2y) \Rightarrow \left(3-2\frac{dy}{dx}\right)\cos(3x-2y) = \left(3-2\frac{dy}{dx}\right)\frac{1}{3x-2y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{2}$$

73. (c)

$$x^4y^5 = 2(x+y)^9 \Rightarrow 4x^3y^5 + 5x^4y^4\frac{dy}{dx} = 18(x+y)^8\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow 4\frac{2(x+y)^9}{x} + 5\frac{2(x+y)^9}{y}\frac{dy}{dx} = 18(x+y)^8\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{4}{x} - \frac{9}{x+y} = \left(\frac{9}{x+y} - \frac{5}{y}\right)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

74. (b)

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{a[\cos\theta - \theta(-\sin\theta) - \cos\theta]}{a[-\sin\theta + \theta\cos\theta + \sin\theta]} = \frac{\theta\sin\theta}{\theta\cos\theta} = \tan\theta .$$

75. (d)

Obviously $x=\cos^{-1}\frac{1}{\sqrt{1+t^2}}$ and $y=\sin^{-1}\frac{t}{\sqrt{1+t^2}}$

$\Rightarrow x=\tan^{-1}t$ and $y=\tan^{-1}t$

$\Rightarrow y=x \Rightarrow \frac{dy}{dx}=1$.

76. (c)

$x=\frac{1-t^2}{1+t^2}$ and $y=\frac{2t}{1+t^2}$

Put $t=\tan\theta$ in both the equations to get

$x=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\cos 2\theta$ and $y=\frac{2\tan\theta}{1+\tan^2\theta}=\sin 2\theta$.

Differentiating both the equations, we get $\frac{dx}{d\theta}=-2\sin 2\theta$ and $\frac{dy}{d\theta}=2\cos 2\theta$.

Therefore $\frac{dy}{dx}=-\frac{\cos 2\theta}{\sin 2\theta}=-\frac{x}{y}$.

77. (d)

$y=\sqrt{x+1+\sqrt{x+1+\sqrt{x+1 ... \text{to } \infty}}} \Rightarrow y=\sqrt{x+1+y}$

$\Rightarrow y^2=x+y+1 \Rightarrow 2y\frac{dy}{dx}=1+\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}(2y-1)=1 \Rightarrow \frac{dy}{dx}=\frac{1}{2y-1}$

78. (b)

$y=(x+1)^{(x+1)^{(x+1)......\infty}} \Rightarrow y=(x+1)^y$

$\Rightarrow \log_e y=y\log_e(x+1)$

$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx}=\frac{y}{(x+1)}+\ln(x+1)\frac{dy}{dx}$

$\Rightarrow \left(\frac{1}{y}-\ln(x+1)\right)\frac{dy}{dx}=\frac{y}{x+1}$

$\Rightarrow (x+1)(1-\ln y)\frac{dy}{dx}=y^2$

79. (a)

$y=x^2+\frac{2}{y} \Rightarrow y^2=x^2y+2$

$\Rightarrow 2y\frac{dy}{dx}=y.2x+x^2\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}=\frac{2xy}{2y-x^2}$

80. (c)

$x = e^{2y+x}$

Taking log both sides, $\log x = (2y + x)\log e = 2y + x$

$\Rightarrow 2y + x = \log x \Rightarrow 2\frac{dy}{dx} + 1 = \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{1-x}{2x}$