LIMITS

EXERCISE - 3

1. $$f(x)=\begin{cases}\dfrac{x}{\sin x} & , \quad x>0\\ 2-x & ; \quad x\le 0\end{cases}$$

$$g(x)=\begin{cases}x+3 & , \quad x<0\\ x^2-2x-2 & ; \quad 1\le x<2\\ x-5 & ; \quad x\ge 2\end{cases}$$

$$g(f(x))=\begin{cases}f(x)+3 & , \quad f(x)<1\\ f^2(x)-2f(x)-2 & ; \quad 1\le f(x)<2\\ f(x)-5 & ; \quad f(x)\ge 2\end{cases}$$

$\boxed{b=t}$

$$=\begin{cases}\dfrac{x}{\sin x}+3 & ; \quad x>0\cap\dfrac{x}{\sin x}<1\\ 2-x+3 & ; \quad x\le 0\cap 2x<1\\ \left(\dfrac{x}{\sin x}\right)^2-2\left(\dfrac{x}{\sin x}\right)-2 & ; \quad x>0\cap 1\le\dfrac{x}{\sin x}<2\\ (2-x)^2-2(2-x)-2 & ; \quad x\le 0\cap 1\le 2-x<2\\ \dfrac{x}{\sin x}-5 & ; \quad x>0\cap\dfrac{x}{\sin x}\ge 2\\ 2-x-5 & ; \quad x\le 0\cap 2-x\ge 2\end{cases}$$

$$=\begin{cases}\phi\\ \phi\\ \left(\dfrac{x}{\sin x}\right)^2-2\left(\dfrac{x}{\sin x}\right)-2 & ; \quad x>0\cap 1\le\dfrac{x}{\sin x}<2\\ \phi\\ \dfrac{x}{\sin x}-5 & ; \quad x>0\cap\dfrac{x}{\sin x}\ge 2\\ 2-x-5 & ; \quad x\le 0\end{cases}$$

$$\Rightarrow g(f(x)) = \begin{cases} -x-3 & ; \quad x \le 0 \\ \left(\frac{x}{\sin x}\right)^2 - 2\left(\frac{x}{\sin x}\right) - 2 & ; \quad 1 \le \frac{x}{\sin x} < 2 \cap x > 0 \\ \frac{x}{\sin x} - 5 & ; \quad x > 0 \cap \frac{x}{\sin x} \ge 2 \end{cases}$$

$\because$ gf(0) = – 0 – 3

= – 3

$$\text{LHL} = \lim_{x\to 0^-} g(f(x))$$

$$= \lim_{x\to 0^-} -(x+3)$$

$$= -0 - 3 = -3$$

$$\text{RHL} = \lim_{x\to 0^+} g(f(x))$$

$$= \lim_{x\to 0^+}\left[\left(\frac{x}{\sin x}\right)^2 - 2\left(\frac{x}{\sin x}\right) - 2\right]$$

$$= \lim_{x\to 0^+}\left[\frac{1}{\left(\frac{\sin x}{x}\right)^2} - \frac{2}{\left(\frac{\sin x}{x}\right)} - 2\right]$$

$$= \lim_{x\to 0^+}\frac{1}{\left(\frac{\sin x}{x}\right)^2} - \lim_{x\to 0^+}\frac{2}{\left(\frac{\sin - x}{x}\right)} - 2$$

$$= 1 - 2 - 2$$

$$= -3$$

$\because$ LHL = RHL = – 3

$$\therefore \lim_{x\to 0} g(f(x)) = -3$$

2. Given $p_n = a^{p_{n-1}} - 1$; $p_{n-1} = a^{p_{n-2}} - 1$

Let $p_1 = a^x - 1$

$$\lim_{x\to 0}\frac{p_n}{x}$$

$$\Rightarrow \lim_{x\to 0}\frac{a^{p_{n-1}} - 1}{x} \times \frac{p_{n-1}}{p_{n-1}}$$

$$\Rightarrow \lim_{x\to 0} \ell na \times \frac{p_{n-1}}{x}$$

$$\Rightarrow \lim_{x\to 0} \ell na \times \frac{a\, p_{n-1} - 1}{x} \times \frac{p_{n-2}}{p_{n-2}}$$

$$\Rightarrow \lim_{x\to 0} (\ell na)^2 \times \frac{p_{n-2}}{x}$$

$$\Rightarrow \lim_{x\to 0} (\ell na)^{n-1} \times \frac{p_1}{x}$$

$$\Rightarrow (\ell na)^{n-1} \times \lim_{x\to 0} \frac{a^x - 1}{x}$$

$$\Rightarrow (\ell na)^n$$

3. $x^3 - (2x + 1)x^2 + (2x - 1)\, x + 1 = 0$ ...(1)

roots of equation a_n, b_n, c_n $a_n < b_n < c_n$

$x = 1$ is a root of equation (1)

so $(x - 1)(x^2 - 2nx - 1) = 0$

$$x = \frac{2n \pm \sqrt{4x^2 + 4.1}}{2.1}$$

$$x = n \pm \sqrt{n^2 + 1}$$

$$x = n \pm \sqrt{n^2 + 1}$$

$$x = n \pm \sqrt{n^2 + 1}$$

$$
\begin{array}{l}
(x-1)\overline{)\,x^3 - (2x+1)x^2 + (2n-1)x + 1} \\
\quad x^3 - x^2 \\
\quad - \;\; + \;\; 1 \\
\hline
\quad -2nx^2 + (2n-1)x + 1 \\
\quad -2nx^2 + 2nx \\
\quad + \qquad - \\
\hline
\qquad -x + 1 \\
\qquad -x + 1 \\
\hline
\end{array}
$$

so the three roots of cubic equation.

$$n + \sqrt{n^2 + 1}, n - \sqrt{n^2 + 1}, 1$$

so $\lim_{n\to\infty} n\, a_n \Rightarrow \lim_{n\to\infty} n(n-\sqrt{n^2+1})\times\frac{n+\sqrt{n^2+1}}{n+\sqrt{n^2+1}}$

$$\Rightarrow \lim_{n\to\infty}\frac{n(n^2-n^2-1)}{n+\sqrt{n^2+1}}$$

$$\Rightarrow \lim_{n\to\infty}\frac{-n}{n\left(1+\sqrt{1+\frac{1}{n^2}}\right)}=-\frac{1}{2}.$$

4. $a_n = 2^2\ [1^2+2^2+3^2+.....n^2]$

$$a_n = 2^2\left[\frac{n(n+1)\ (2n+1)}{6}\right]$$

$$a_n + b_n = \frac{2n(2n+1)\ (4n+1)}{6}$$

$$b_n = \frac{2n(2n+1)\ (4n+1)}{6} - \frac{4(n(n+1))\ (2n+1)}{6}$$

$$b_n = \frac{n(2n+1)}{6}[2(4n+1)-4(n+1)]$$

$$b_n = \frac{n(2n+1)}{6}\ [4n-2] = \frac{n(2n+1)(2n-1)}{3}$$

$$\lim_{n\to\infty}\frac{\sqrt{a_n}-\sqrt{b_n}}{\sqrt{n}}$$

$$\lim_{n\to\infty}\frac{\sqrt{\frac{2^n n(n+1)\ (2n+1)}{6}}}{\sqrt{n}}-\frac{\sqrt{\frac{n(n+1)\ (2n+1)}{2}}}{\sqrt{n}}$$

$$\lim_{n\to\infty}\frac{1}{\sqrt{3}}\left(\sqrt{4n^2+6n+2}-\sqrt{4n^2-1}\right)$$ by ratio radization

$$\lim_{n\to\infty} = \frac{\frac{1}{\sqrt{3}}(6n+2+1)}{\sqrt{4n^2+6n+2}+\sqrt{4n^2-1}}$$

$$\lim_{n\to\infty}\frac{1}{\sqrt{3}}\frac{6+\frac{3}{n}}{\sqrt{4+\frac{6}{n}+\frac{2}{n^2}}+\sqrt{\frac{4n^2-1}{n^2}}}$$

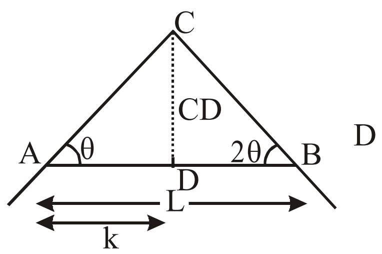
$$=\frac{1}{\sqrt{3}}\cdot\frac{6}{2+2}$$

$$=\frac{\sqrt{3}}{2}$$

5. $\tan\theta = \dfrac{CD}{x}$

$\tan 2\theta = \dfrac{CD}{L-x}$

$x\tan\theta = (L-x)\,\dfrac{2\tan\theta}{1-\tan^2\theta}$



$x = \dfrac{2(L-x)}{1-\tan^2\theta}$

$x(1-\tan^2\theta) = 2L - 2x$

$x(2+1-\tan^2\theta) = 2L$

$x = \dfrac{2L}{3-\tan^2\theta}$, $\tan\theta \to 0$

$\theta \to 0$ gives $\boxed{x = \dfrac{2L}{3}}$

6. Put $\dfrac{x}{1+x} = y \Rightarrow y \to 1$ as $x \to \infty$

$1 - \dfrac{1}{1+x} = y \Rightarrow 1 + x = \dfrac{1}{1-y} \Rightarrow x = \dfrac{1}{1-y} - 1 \Rightarrow x = \dfrac{y}{1-y}$

$$e^{\lim\limits_{y\to 1}\left[\cos 2\pi y^a - 1\right]\left(\frac{y}{1-y}\right)^2} = e^{\lim\limits_{y\to 1}(-2\sin^2 \pi y^a)\left(\frac{y^2}{(1-y)^2}\right)}$$

$$e^{\lim\limits_{y\to 1}-2\left[\frac{\sin^2\pi(1-y^a)}{\pi^2(1-y^a)^2}\cdot\frac{\pi^2(1-y^a)^2}{(1-y)^2}\cdot y^2\right]}$$

$$e^{-2\pi^2\left(\frac{1-y^a}{1-y}\right)^2 \cdot y^2 = e^{-2\pi^2 a^2}} \qquad \left[\because \lim_{x\to 1}\frac{x^n-1}{x-1}=n\right]$$

Alt.: Let limit = e^L

$$\text{Let} = \frac{x}{1+x} = t$$

$$L = \lim_{t\to 1}\left[\frac{\cos(2\pi-2\pi t^a)-1}{(2\pi(1-t^a))^2}\right]\frac{t^2}{(1-t)^2}\frac{4\pi^2(1-t^a)^2}{a^2}$$

$$= \left(\frac{-1}{2}\right)4\pi^2(a^2) \quad e^L = e^{-2\pi^2 a^2}$$

7. $$\lim_{x\to 0}\left(\frac{x-1+\cos x}{x}\right)^{\frac{1}{x}}$$

$$\lim_{x\to 0}\left(\frac{x-1+1-\frac{x^2}{2!}+\frac{x^4}{4!}+.....}{x}\right)^{1/x}$$

$$\lim_{x\to 0}\left(1-\frac{x}{2}+\frac{x^3}{4!}.........\right)^{1/x} \Rightarrow e^{\lim_{x\to 0}\left(\frac{-x}{2}+\frac{x^2}{4!}+.........\right)\frac{1}{x}} = e^{-1/2}$$

Alt: $e^{\lim_{x\to 0}\left(\frac{x-1+\cos x - x}{x}\right)\frac{1}{x} \Rightarrow e^{-1/2}}$

$$e^{\lim_{x\to 0}\frac{-2\sin^2\frac{x}{2}}{\frac{4x^2}{4}} = e^{-1/2}}$$

8. $$\lim_{x\to\infty}\left(\frac{a_1^{\left(\frac{1}{x}\right)}+a_2^{\left(\frac{1}{x}\right)}+a_3^{\left(\frac{1}{x}\right)}+.....+a_n^{\left(\frac{1}{x}\right)}}{n}\right)^{nx}$$

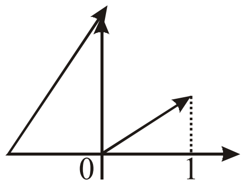
$$e^{\lim_{x\to\infty}}\frac{a_1^{1/x}-1}{1/x}+\frac{a_2^{1/x}-1}{1/x}+....+\frac{a_n^{1/x}-1}{1/x}$$

$$e^{(\ell n a_1)+(\ell n a_2)+...+(\ell n a_n)} = e^{\ell n(a_1.a_2......a_n)}$$

$= a_1.a_2.......a_n.$

9. $\lim_{x\to 0^+}\frac{\sin^{-1}(1-\{x\})\cos^{-1}(1-\{x\})}{\sqrt{2}\{x\}.(1-\{x\})}$

$\{x\} \to 0$ for $x \to 0^+$



$\lim_{x\to 0^+}\frac{\sin^{-1}(1-\{x\})\sin^{-1}\sqrt{1-(1-\{x\})^2}}{\sqrt{2}\{x\}(1-\{x\})}$ $[1 - 1 - \{x\}^2 + 2\{x\}]$

$\lim_{x\to 0^+}\frac{\sin^{-1}(1-\{x\})\sin^{-1}\sqrt{\{x\}(2-\{x\})}}{\sqrt{2}\{x\}(1-\{x\})} = \frac{\pi}{2}$

$\lim_{x\to 0^-}\frac{\sin^{-1}(1-\{x\})(\cos^{-1}(1-\{x\}))}{\sqrt{2}\{x\}(1-\{x\})}[\{x\}\to 1]$

$= \frac{\pi}{2\sqrt{2}}$

10. $\lim_{x\to a}\frac{1}{(a^2-x^2)^2}\left[\frac{a^2+x^2}{ax}+\cos\frac{\pi}{2}(a+x)-\cos\left(\frac{\pi}{2}(a-x)\right)\right]$

$\lim_{x\to a}\frac{1}{(a^2-x^2)^2}\left[\frac{(a-x)^2}{ax}+\cos\frac{\pi}{2}(a+x)+(1+1)\frac{2ax}{ax}-\cos\frac{\pi}{2}(a-x)\right]$

$\lim_{x\to a}\frac{1}{(a+x)^2ax}+\frac{\left\{\cos\frac{\pi}{2}(a+x)\right\}ax}{(a^2-x^2)^2ax}+2ax$

$\lim_{x\to a}\frac{1}{(a+x)^2ax}+\frac{2\cos^2\frac{\pi}{4}(a+x)}{(a^2-x^2)^2}+\frac{2\sin^2\frac{\pi}{4}(a-x)}{(a^2-x^2)^2}$

$\frac{1}{4a^4}+\lim_{x\to a}\frac{4\sin^2\frac{\pi}{4}(a-x)}{(a^2-x^2)^2}$ $\left[\cos\left(\frac{a\pi}{4}+\frac{\pi}{4}x\right)=\sin\left(\frac{a\pi}{2}-\left(\frac{(a+x)\pi}{4}\right)\right)\right]+$

$$\lim_{x\to a}\frac{4\sin^2\frac{\pi}{4}(a-x)\frac{\pi^2}{16}}{\frac{(a-x)^2\pi^2}{16}(a+x)^2}$$

$$=\frac{1}{4a^4}+\frac{\pi^2}{16a^2}=\frac{a^2\pi^2+4}{16a^4}.$$

11. $$L=\lim_{x\to 1}\frac{(1-x)(1-x^2)\,(1-x^3)....(1-x^{2n})}{[(1-x)\,(1-x^2)\,(1-x^3).....(1-x^n)]^2}$$

$$\frac{1-x^{n+1}}{1-x}.\frac{1-x^{n+2}}{1-x^2}.\frac{1-x^{n+3}}{1-x^3}+.......\frac{(1-x^{2n})}{(1-x^n)}$$

divide by (1 – x) in N^r and D^r both→

$$\frac{(n+1)\,(n+2).......2n}{1.2.3....n}=\frac{2n!}{n!n!}={}^{2n}C_n$$

(a) (b) $\frac{2.6.10.....(4n-2)}{n!}=\frac{2^n[1.3.5.....(2n-1)]}{n!}$

$$=\frac{2^n[1.3.5.....2n-1]\,[2.4.6......2n]}{n!.2^n.n!}=\frac{2n!}{n!n!}$$

(c) ${}^{2n-1}C_n+{}^{2n-1}C_{n+1}$

(d) ${}^{2n}C_n$

12. $\lim_{x\to 1}\frac{1-x+\ell nx}{1+\cos\pi x}$ $\qquad\lim_{h\to 0}\frac{h+\ell n(1-\ell h)/\pi^2h^2}{\frac{1-\cos\pi\ell h}{\pi^2\ell h^2}}$

put x = 1 – h

$$\lim_{\ell h\to 0}\frac{h+\ell n(1-\ell h)}{1+\cos\pi(1-\ell h)}$$

$$\lim_{\ell h\to 0}\frac{h+\ell h(1-\ell h)}{2\sin^2\frac{\pi\ell h}{2}}$$

expanding log (1 – ℓh) →

$$\lim_{\ell h\to 0}\frac{\ell h+\left[-h-\frac{h^2}{2}-\frac{h^3}{3}......\right]}{2\sin^2\left(\frac{\pi\ \ell h}{2}\right)}=\lim_{\ell h\to 0}\frac{-\frac{h^2}{2}\frac{\pi^2}{2}}{\frac{\pi^2}{2}.2\sin^2\left(\frac{\pi h}{2}\right)}=-\frac{1}{\pi^2}$$

13. $\lim_{y\to 0}\left[\lim_{x\to\infty}\frac{\exp.\left(x\ell n\left(1+\frac{ay}{x}\right)\right)-\exp.\left(x\ell n\left(1+\frac{by}{x}\right)\right)}{y}\right]$

$\lim_{y\to 0}\left[\lim_{x\to\infty}\frac{\left(1+\frac{ay}{x}\right)^{x}-\left(1+\frac{by}{x}\right)^{x}}{y}\right]$

$\lim_{y\to 0}\frac{e^{ay}-e^{by}}{y}=\lim_{y\to 0}\frac{[e^{(a-b)y}-1]}{y}.e^{by}$

$=a-b$

14. $x_1=\sqrt{2+2\cos\frac{\pi}{6}}=2\cos\frac{\theta}{2}$ if $\theta=\frac{\pi}{6}$

$x_2=2\cos\frac{\theta}{4}$ .... $x_n=2\cos\frac{\theta}{2^n}$

$\lim_{n\to\infty}2^{(n+1)}\sqrt{2-2\cos\frac{\theta}{2^n}}=\lim_{n\to\infty}2^{n+1}2\sin\frac{\theta}{2^{n+1}}$

$=\lim_{n\to\infty}\frac{2\left(\sin\frac{\theta}{2}n+1\right)\theta}{\frac{\theta}{2^{n+1}}}=2\theta=2\times\frac{\pi}{6}=\frac{\pi}{3}$

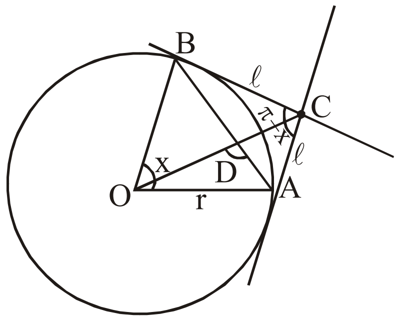
15. $\lim_{x\to 0}\frac{\ell n(1+x)^{1+x}}{x^2}-\frac{1}{x}$

$\frac{\lim_{x\to 0}\frac{(1+x)}{x}\left[\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}......\right]-1}{x}$

$\lim_{x\to 0}\frac{(1+x)\left[1-\frac{x}{2}+\frac{x^2}{3}.....\right]-1}{x}=\frac{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}.....\right)+x\left(1-\frac{x}{2}.....\right)}{x}=-\frac{1}{2}+1=\frac{1}{2}$

16. **Act:** ar (ΔABC) = $\frac{1}{2}\ell^2 \sin(\pi = x)$

ar (ΔABC) = $\frac{1}{2}r^2x - \frac{1}{2}r^2 \sin x$



$\ell = r \tan \frac{x}{2}$

$$\frac{T(x)}{S(x)} = \frac{\frac{1}{2}\tan^2\frac{x}{2}\sin x}{\frac{1}{2}(x - \sin x)}$$

(a) ar(ΔABC) = T(x) = $\frac{1}{2}$.AB.CD = $\frac{1}{2}$.2r sin $\frac{x}{2}$. $\left(r\sec\frac{x}{2} - r\cos\frac{x}{2}\right)$

$$= r^2 \sin\frac{x}{2}\frac{\sin^2\frac{x}{2}}{\cos\frac{x}{2}} = r^2\frac{\sin^3\frac{x}{2}}{\cos\frac{x}{2}} = \frac{1}{2}\tan^2\frac{x}{2}\sin x \quad (\because r = 1)$$

(b) S(x) = $\frac{1}{2}x - \frac{1}{2}\sin x = \frac{1}{2}(x - \sin x)$

(area of ar c = $\frac{1}{2}r^2\theta$| area of $\Delta = \frac{1}{2}$b C sin A)

(c) $\lim\limits_{x\to 0}\frac{\frac{1}{2}\tan^2\frac{x}{2}.\sin x}{\frac{1}{2}(x - \sin x)}$ $\lim\limits_{x\to 0}\frac{\tan^2\frac{x}{2}.\sin x}{x - \left(x - \frac{x^3}{3!} + ...\right)}$

$$\Rightarrow \lim_{x\to 0}\frac{\tan^2\frac{x}{2}}{4\frac{x^2}{4}}.\frac{\sin x}{x\frac{1}{3!}} = \frac{6}{4} = \frac{3}{2}$$

Alternate-

$$\frac{\dfrac{\frac{1}{2}\tan^2\frac{x}{2}.\sin x}{4\quad\frac{x^2}{4}\quad x}}{\frac{1}{2}\frac{x - \sin x}{x^3}} = \frac{\frac{1}{4}}{\frac{1}{6}} = \frac{3}{2}$$

17. $\sin^3 x = \frac{3\sin x - 4\sin^3 x}{4}$ $\quad\quad$ $\sin 3x = 3\sin x - 4\sin^3 x$

$$\frac{1}{4}\sum_{n=1}^{n}\left(3\sin\frac{x}{3} - \sin x\right) + 3\left(3\sin\frac{x}{3^2} - \sin^3\frac{x}{3^2}\right) + 3^2\left(3\sin\frac{x}{3^3} - \sin^3\frac{x}{3^3}\right)$$

$$\frac{1}{4}\sum_{n=1}^{n} 3^{n-1}\sin.\frac{x}{3^{n-1}} - \sin x$$

$$\frac{1}{4}\left[\underset{n\to\infty}{\mathrm{Lim}}\frac{x\sin\left(\frac{x}{3^n}\right)}{\frac{x}{3^n}} - \sin x\right]$$

$$\frac{1}{4}[x - \sin x]$$

$$= g(x) = x - 4\left[\frac{1}{4}(x - \sin x)\right]$$

$g(x) = \sin x$.

18. $f(n,0) = \prod_{r=1}^{n}\left(1 - \tan^2\frac{\theta}{2^r}\right)$ $\quad\quad$ $\left|\left(\text{use}\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}\right)\right.$

$$f(n, \theta) = \left(1 - \tan^2\frac{\theta}{2}\right)\left(1 - \tan^2\frac{\theta}{2^2}\right)......\left(1 - \tan^2\frac{\theta}{2^n}\right)$$

$$= \frac{2\tan\frac{\theta}{2}}{\tan\theta}.\frac{2\tan\frac{\theta}{2^2}}{\tan\frac{\theta}{2}}.\frac{2\tan\frac{\theta}{2^3}}{\tan\frac{\theta}{2^2}}........$$

$$f(n, \theta) = \frac{2^n\tan\frac{\theta}{2^n}}{\tan\theta}$$

$$\underset{n\to\infty}{\mathrm{Lim}}\frac{\theta\tan\frac{\theta}{2^n}}{\tan\theta\frac{\theta}{2^n}} = \frac{\theta}{\tan\theta} .$$

19. $$\underset{x\to 0}{\mathrm{Lim}}\frac{\sqrt{\frac{\cos 2x + (1+3x)^{1/3}}{2}} - \sqrt[3]{\cos^3 x - \ln(1+x)}}{x}$$

By L'Hospital's Rule

$$\lim_{x\to 0}\left\{\frac{\dfrac{-2\sin 2x+(1+3x)^{-2/3}}{2}}{2\sqrt{\dfrac{\cos 2x+(1+3x)^{1/3}}{2}}}-\frac{-3\cos^2 x\sin x-\dfrac{1}{1+x}}{3\left(\cos^3 x-\ln(1+x)\right)^{-2/3}}\right\}=\frac{1}{4}+\frac{1}{3}=\frac{7}{12}$$

Hence $a+b=19$.

20. $$e^t=1+t+\frac{t^2}{2!}+\frac{t^3}{3!}.......$$

put $\frac{\pi}{x}=t$, $$e^{-t}=1-t+\frac{t^2}{2!}-\frac{3}{3!}+......$$

$$\lim_{t\to 0}\left(\frac{e^t+e^{-t}}{2\cos t}\right)^{\pi^2/t^2}$$

$$e^{\lim\limits_{t\to 0}\left(\frac{e^t+e^{-t-2+2}-2\cos t}{2\cos t}\right)\left(\frac{\pi^2}{t^2}\right)}$$

$$e^{\lim\limits_{t\to 0}\left[\frac{2+2t^2/2!+2\frac{t^4}{4!}+......-2\cos t}{2\cos t}\right]\left(\frac{\pi^2}{t^2}\right)}$$

$$e^{\lim\limits_{t\to 0}\frac{2\pi^2}{2!}+......+\frac{4\sin^2 t/2}{4t^2/4}.\pi^2 \Big/ 2\cos t}$$

$$e^{\frac{\pi^2+\pi^2}{2}-e^{\pi^2}}$$

Alternate

$$e^{\lim\limits_{t\to 0}\left(\frac{e^t+e^{-t}-2+2(1-\cos t)}{2\cos t}\right)\frac{\pi^2}{t^2}}$$

$$=e^{\frac{\pi^2}{2}\lim\limits_{t\to 0}\frac{e^t+e^{-t}-2}{t^2}+\frac{4\sin^2 t/2}{t^2}}$$

$$=e^{\pi^2/2(1+1)}=e^{\pi^2}$$

21.

when $P\to A$
$T\to A$
$\pi-2\theta\to 0$

$$\frac{AQ}{\sin(\pi-\theta)}=\frac{\ell}{\sin\left(\frac{\pi}{2}-\theta\right)}$$

$AQ = \dfrac{\ell \sin\theta}{\cos\theta}$ ...(1)

$AP = \ell = 2r \cos\left(2\theta - \dfrac{\pi}{2}\right) = 2r \sin 2\theta$ when $P \to A$

$AQ = \dfrac{2r\sin 2\theta.\sin\theta}{\cos\theta} = 4r\sin^2\theta$ $\theta \to \dfrac{\pi}{2}$

Alt :- gemetrically:-

$AQ = AS + SQ$

in $\Delta TAQ \Rightarrow \angle Q = \dfrac{0}{2}$

in $\Delta PSQ \to$

$\angle Q = \angle P = \dfrac{Q}{2}$

$SQ = PS$

as $P \to A \Rightarrow SP = SA = 2r = SQ \Rightarrow AQ = AS + SQ = 2r + 2r = 4r$

22. $L = \lim\limits_{x\to 0}\left[\dfrac{1}{\ell n(1+x)} - \dfrac{1}{\ell n(x+\sqrt{1+x^2})}\right]$

put x = – x ($\because$ x → 0, we can put x = – x)

$L = \lim\limits_{x\to 0}\left[\dfrac{1}{\ell n(1-x)} - \dfrac{1}{\ell n(\sqrt{1+x^2-x})}\right]$

Adding both:

$$2L = \lim_{x\to 0}\frac{1}{\ell n(1+x)} + \frac{1}{\ell n(1-x)} - \frac{1}{\ell n(x+\sqrt{1+x^2})} - \frac{1}{\ell n(\sqrt{(1+x^2-x)})}$$

$$2L = \lim_{x\to 0}\frac{\ell n(1-x^2)}{\ell n(1+x)\ell n(1-x)} - \left\{\frac{\ell n(1+x^2-x^2)}{\ell n(x+\sqrt{1+x^2})\ell n(\sqrt{1+x^2}-x)}\right\} = 0 \; (\because N_r \text{ is absolute zero})$$

$$= \lim_{x\to 0}\frac{\ell n(1-x^2)}{-x^2} \Big/ \frac{\ell n(1+x)}{x}.\frac{\ell n(1-x)}{(-x)} = +1$$

$L = +\dfrac{1}{2}$

$$\frac{+\frac{1}{2}+153}{+\frac{1}{2}} = \frac{306+1}{+1} = 307$$

23 Let $U_n = \lim\limits_{x\to 0} \frac{1-\cos 3x\cdot\cos 3^2x\cdot\cos 3^3x.........\cos 3^n x}{x^2}$

and $V_n = \lim\limits_{x\to 0} \frac{1-\cos\frac{x}{3}\cdot\cos\frac{x}{3^2}\cdot\cos\frac{x}{3^3}.........\cos\frac{x}{3^n}}{x^2}$

$U_n = \lim\limits_{x\to 0} \frac{-D(\cos 3x\cdot\cos 3^2x\cdot\cos 3^3x.........\cos 3^n x)}{2x}$

now let $y = \cos 3x \cdot \cos 3^2x \cdot \cos^3 x .... \cos 3^n x$

$\ln y = \ln\cos 3x + \ln\cos 3^2x + ... + \ln\cos 3^n x$

$$\frac{1}{y}\frac{dy}{dx} = -[3\tan 3x + 3^2\tan 3^2x + ... + 3^n\tan 3^n x]$$

$$\frac{dy}{dx} = -\prod_{r=1}^{n}\cos 3^r x[3\tan 3x + 3^2\tan 3^2x + ... + 3^n\tan 3^n x]$$

$$\therefore\quad U_n = \lim_{x\to 0}\frac{3\tan 3x + 3^2\tan 3^2x + ... + 3^n\tan 3^n x}{2x} = \frac{3^2+(3^2)^2+(3^3)^2+......+(3^n)^2}{2}$$

$$U_n = \frac{3^2[3^{2n}-1]}{(3^2-1)\cdot 2} \qquad ....(1)$$

|||ly replacing 3^r by $\frac{1}{3^r}$ we get

$$V_n = \frac{\frac{1}{3^2}\left[1-\frac{1}{3^{2n}}\right]}{\left(1-\frac{1}{3^2}\right)\cdot 2} = \frac{(3^{2n}-1)}{3^{2n}(3^2-1)\cdot 2} \qquad ....(2)$$

$$\therefore\quad \frac{U_n}{V_n} = 3^{2n+2} = 3^{10} \text{ (given)}$$

$\therefore\quad 2n + 2 = 10 \quad\Rightarrow\quad \boxed{n = 4}$

24. (a) $\lim\limits_{x\to 0^+}\left(p_1a_1^x + p_2a_2^x + ... + p_na_n^x\right)^{1/x} \to 1^\infty$ form as $\lim\limits_{x\to 0^+} a_i^x = 1$ & $\sum\limits_{i=1}^{n} p_i = 1$

$$\Rightarrow \lim = e^{\lim\limits_{x\to0^+}\left(\frac{p_1a_1^x+p_2a_2^x+.......+p_na_n^x-1}{x}\right)} \quad \text{by } \lim_{x\to0}\left(f(x)\right)^{g(x)} = e^{\lim\limits_{x\to0} g(x)(f(x)-1)}$$

Now $1 = p_1 + p_2 + p_3 + ...p_n$

$$\Rightarrow \lim = e^{\lim\limits_{x\to0^+}\left(\frac{p_1a_1^x+p_2a_2^x+.......+p_na_n^x-(p_1+p_2+p_3+...p_n)}{x}\right)}$$

$$\Rightarrow \lim = e^{\lim\limits_{x\to0^+}\left(p_1\left(\frac{a_1^x-1}{x}\right)+p_2\left(\frac{a_2^x-1}{x}\right)+.......+p_n\left(\frac{a_n^x-1}{x}\right)\right)}$$

$$\Rightarrow \lim = e^{p_1\ln a_1+p_2\ln a_2+.......+p_n\ln a_n} = a_1^{p_1}a_2^{p_2}a_3^{p_3}...a_n^{p_n} \quad .$$

(b) $\lim\limits_{x\to\infty} F(x) = L_2 = \lim\limits_{x\to\infty}\left(p_1a_1^x + p_2a_2^x + ... + p_na_n^x\right)^{1/x}$ (∞^0 form)

[only when a_1 a_2 etc. > 1]

$\therefore \quad \ln L_2 = \lim\limits_{x\to\infty}\dfrac{\ln\left(p_1a_1^x + p_2a_2^x + ... + p_na_n^x\right)}{x}$

using L'Hospital's Rule

$$L_2 = \lim_{x\to\infty}\frac{\left(p_1\ln a_1\, a_1^x + p_2\ln a_2\, a_2^x + ... + p_n\ln a_n\, a_n^x\right)}{p_1a_1^x + p_2a_2^x + ... + p_na_n^x} \qquad ....(1)$$

dividing by a_1^x and taking limit, we get

$\lim\limits_{x\to\infty}, \left(\dfrac{a_2}{a_1}\right)^x, \left(\dfrac{a_3}{a_2}\right)^x$, etc all vanishes as $x \to \infty$

$= \dfrac{p_1\ln a_1}{p_1} = \ln a_1$

hence $\ln L_2 = \ln a_1 \quad \Rightarrow \quad L_2 = a_1$ **Ans.**

(c) $\lim\limits_{x\to-\infty} F(x) = L_3$ (say)

$\therefore \quad \ln L_3 = \lim\limits_{x\to-\infty}\dfrac{\left(p_1\ln a_1\, a_1^x + p_2\ln a_2\, a_2^x + ... + p_n\ln a_n\, a_n^x\right)}{p_1a_1^x + p_2a_2^x + ... + p_na_n^x}$

dividing by $(a_n)^x$ and taking $\lim\limits_{x\to-\infty}, \left(\dfrac{a_1}{a_n}\right)^x, \left(\dfrac{a_2}{a_n}\right)^x$ etc vanishes

$\therefore \quad \ln L_3 = \dfrac{p_n\ln a_n}{p_n} \quad \Rightarrow \quad L_3 = a_n$

25. $\displaystyle\sum_{r=1}^{n}\cot^{-1}\left(r^2+\frac{3}{4}\right) = \sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{\frac{3}{4}+r^2}\right) = \sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{1+r^2-\frac{1}{4}}\right)$