LIMITS
EXERCISE 2(A)

1. (A)(D)

put $\theta = -1$; $\frac{1-1-2}{2} \le f(-1) \le \frac{1-2-1}{2}$

$-1 \le f(-1) \le -1 \quad \Rightarrow \quad f(-1) = -1$

$$\lim_{\theta\to-1} \frac{\theta^2+\theta-2}{\theta+3} = -1 = \lim_{\theta\to-1} \frac{\theta^2+2\theta-1}{\theta+3}$$

using sequeeze play theorem

$$\lim_{\theta\to-1} \frac{f(\theta)}{\theta^2} = -1; \ \lim_{\theta\to-1} f(\theta) = -1.$$

2. (A)(B)(C)(D)

$$l_1 = \lim_{x\to\infty} \sqrt{\frac{1-\frac{\cos^2 x}{x}}{1+\frac{\sin x}{x}}} = 1$$

$$l_2 = \lim_{h\to0^+} 2\int_0^1 \frac{h\,dx}{h^2+x^2} = \lim_{h\to0^+} \left[2\frac{h}{h}\tan^{-1}\frac{x}{h}\right]_{0^+}^1 = \pi \ \textbf{Ans.}$$

Note: $\frac{22}{7} > \pi \qquad \left[\frac{22}{7} = 3.1428571 \text{ and } \pi \approx 3.1415929\right]$.

3. (B)(C)

(A)

$$\lim_{x\to3^+}\left(\left[[x]\right]-[2x+1]\right) = 3-7 = -4$$

$$\lim_{x\to3^-}\left(\left[[x]\right]-[2x+1]\right) = 2-6 = -4$$

$$\lim_{x\to3^+} f(x) = \lim_{x\to3^-} f(x)$$

Hence limit exists.

(B)

$$\lim_{x\to1^+}([x]-x) = 1-1 = 0$$

$$\lim_{x\to1^-}([x]-x) = 0-1 = -1$$

$$\lim_{x\to1^+} f(x) \ne \lim_{x\to1^-} f(x)$$

Limit does not exist

(C)

$$\lim_{x\to0^+}\left(\{x\}^2-\{-x\}^2\right) = 0-1 = -1$$

$$\lim_{x\to0^-}\left(\{x\}^2-\{-x\}^2\right) = 1-0 = 1$$

$$\lim_{x\to1^+} f(x) \ne \lim_{x\to1^-} f(x)$$

Limit does not exist

(D)

$$\lim_{x\to0^+} \frac{\tan(\operatorname{sgn}(x))}{\operatorname{sgn}(x)} = \frac{\tan 1}{1} = \tan 1$$

$$\lim_{x\to0^-} \frac{\tan(\operatorname{sgn}(x))}{\operatorname{sgn}(x)} = \frac{\tan(-1)}{-1} = \tan 1$$

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^-} f(x)$$

Hence limit exists.

4. (A)(C)(D)

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} \frac{\tan^2\{x\}}{x^2-[x]^2} = \lim_{x\to0^+} \frac{\tan^2 x}{x^2} = 1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sqrt{\{x\}\cot\{x\}} = \lim_{x\to 0^+} \sqrt{\frac{x+1}{\tan(x+1)}} = \sqrt{\cot 1}$$

5. (B)(D)

(A) $\lim_{x\to\infty} x^{\frac{1}{4}} \sin\frac{1}{\sqrt{x}} = \lim_{y\to 0^+} \frac{\sin\sqrt{y}}{y^4} = \infty$

(B) $\lim_{x\to\frac{\pi}{2}} (1-\sin x)\tan x = \lim_{x\to\frac{\pi}{2}} \left(\frac{1-\sin x}{\cos x}\right)\sin x = \lim_{x\to\frac{\pi}{2}} \left(\frac{\cos^2 x}{\cos x}\right)\frac{\sin x}{1+\sin x} = 0$

(C) $\lim_{x\to\infty} \left(\frac{2x^2+3}{x^2+x-5}\right)\operatorname{sgn}(x) = \lim_{x\to\infty} \left(\frac{2+\frac{3}{x^2}}{1+\frac{1}{x}-\frac{5}{x^2}}\right) = 2$

(D) $\lim_{x\to 3^+} \frac{[x]^2-9}{x^2-9} = 0$

6. (A)(B)

$$f(x) = \lim_{n\to\infty} \frac{2x^{2n}\sin\frac{1}{x}+x}{1+x^{2n}}$$

for $x<1,\ f(x) = \lim_{n\to\infty} \frac{2x^{2n}\sin\frac{1}{x}+x}{1+x^{2n}} = x$ $\quad\left\{\lim_{n\to\infty} x^{2n} = 0 \text{ if } x<1\right\}$

for $x=1,\ f(x) = 2\sin\frac{1}{x}+x$ $\quad\left\{\lim_{n\to\infty} x^{2n} = 1 \text{ if } x=1\right\}$

for $x>1,\ f(x) = \lim_{n\to\infty} \frac{2\sin\frac{1}{x}+\frac{1}{x^{2n-1}}}{\frac{1}{x^{2n}}+1} = 2\sin\frac{1}{x}$ $\quad\left\{\lim_{n\to\infty} \frac{1}{x^{2n}} = 0 \text{ if } x>1\right\}$

Now

(A) $\lim_{x\to\infty} xf(x) = \lim_{x\to\infty} 2x\sin\frac{1}{x} = 2\lim_{y\to 0}\frac{\sin y}{y} = 2$

(B) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2\sin\frac{1}{x} = 2\sin 1$ & $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1$

$\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$, hence $\lim_{x\to 1} f(x)$ does not exist

(C) $\lim_{x\to 0} f(x) = \lim_{x\to 0} x = 0$

(D) $\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} x = -\infty$

7. (A)(B)(D)

$$\lim_{x\to\infty}\left(\sqrt{x^4+ax^3+3x^2+bx+2}-\sqrt{x^4+2x^3-cx^2+3x-d}\right)$$

$$= \lim_{x\to\infty}\left(\frac{x^4+ax^3+3x^2+bx+2-\left(x^4+2x^3-cx^2+3x-d\right)}{\sqrt{x^4+ax^3+3x^2+bx+2}+\sqrt{x^4+2x^3-cx^2+3x-d}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{(a-2)x^3+(c+3)x^2+(b-3)x+d+2}{\sqrt{x^4+ax^3+3x^2+bx+2}+\sqrt{x^4+2x^3-cx^2+3x-d}}\right)$$

$$= \lim_{y\to 0}\left(\frac{\left((a-2)+(c+3)y+(b-3)y^2+(d+2)y^3\right)}{y\sqrt{1+ay+3y^2+by^3+y^4}+\sqrt{1+2y-cy^2+3y^3-dy^4}}\right)$$

Clearly a = 2. Now

$$\text{Lim} = \lim_{y\to 0}\left(\frac{(c+3)+(b-3)y+(d+2)y^2}{\sqrt{1+ay+3y^2+by^3+y^4}+\sqrt{1+2y-cy^2+3y^3-dy^4}}\right)=\frac{c+3}{2}$$

Hence $\frac{c+3}{2}=4$ or c = 5.

Also $b,d\in R$.

8. Options need correction

$$\lim_{x\to 0}\frac{a+b\sin x-\cos x+ce^x}{x^3}=\lim_{x\to 0}\frac{a+b\left(x-\frac{x^3}{3!}+...\right)-\left(1-\frac{x^2}{2}+...\right)+c\left(1+x+\frac{x^2}{2}+\frac{x^3}{3!}+...\right)}{x^3}$$

$$= \lim_{x\to 0}\frac{(a-1+c)+(b+c)x+\left(c+\frac{1}{2}\right)x^2+\frac{(c-b)}{6}x^3}{x^3}$$

Clearly $a-1+c=0, b+c=0 \ \& \ c+\frac{1}{2}=0$

$\Rightarrow c=-\frac{1}{2}, b=\frac{1}{2}, a=\frac{3}{2}$.

9. (C)

$$\lim_{x\to 0^+}\frac{a^{[x]+x}}{[x]+x}=\lim_{x\to 0^+}\frac{a^x}{x}=\infty \quad \& \quad \lim_{x\to 0^-}\frac{a^{[x]+x}}{[x]+x}=\lim_{x\to 0^-}\frac{a^{x-1}}{x-1}=-\frac{1}{a}$$

$\lim_{x\to 0^+} f(x)\neq \lim_{x\to 0^-} f(x)$.

Limit does not exist.

10. (A)(B)

(A) $\lim_{x\to\infty}\left(\frac{x}{2+x}\right)^{2x}=e^{\lim_{x\to\infty}2x\left(\frac{x}{2+x}-1\right)}$ $\{1^\infty \ \text{form}\}$

$$= e^{-\lim_{x\to\infty}\left(\frac{4x}{2+x}\right)} = e^{-\lim_{x\to\infty}\left(\frac{4}{\frac{2}{x}+1}\right)} = e^{-4}$$

(B) $\lim_{x\to 1}\left(\frac{x}{2+x}\right)^{2x} = \left(\frac{1}{2+1}\right)^2 = \frac{1}{9}$

11. (A)(B)(C)

$$\lim_{x\to 0^+}\frac{ae^{1/x}+be^{-1/x}}{ce^{1/x}+de^{-1/x}} = \lim_{x\to 0^+}\frac{a+be^{-2/x}}{c+de^{-2/x}} = \frac{a}{c}$$

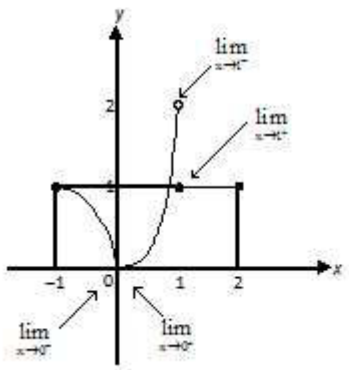
$$\lim_{x\to 0^-}\frac{ae^{1/x}+be^{-1/x}}{ce^{1/x}+de^{-1/x}} = \lim_{x\to 0^+}\frac{ae^{2/x}+b}{ce^{2/x}+d} = \frac{b}{d}$$

Hence a = 2c, b = 2d

Now $bx^2 + (a-2c)x - 2d = 0 \Rightarrow x^2 = 1$.

12. (B)(C)(D)

Clear from the figure.



13. (A)(B)(D)

Conceptual question.

14. Options need correction

$$\lim_{x\to a}\left(2-\frac{a}{x}\right)^{\tan\frac{\pi x}{2a}} = e^{\lim_{x\to a}\tan\frac{\pi x}{2a}\left(2-\frac{a}{x}-1\right)} \quad \{1^\infty \text{ form}\}$$

$$= e^{\lim_{y\to 0}\tan\frac{\pi(a+y)}{2a}\left(1-\frac{a}{a+y}\right)} = e^{\lim_{y\to 0}\tan\left(\frac{\pi}{2}+\frac{\pi y}{2a}\right)\left(\frac{y}{a+y}\right)}$$

$$= e^{-\lim_{y\to 0}\cot\frac{\pi y}{2a}\left(\frac{y}{a+y}\right)} = e^{-\lim_{y\to 0}\frac{\frac{\pi y}{2a}}{\tan\frac{\pi y}{2a}}\frac{2}{\pi(a+y)}}$$

$$= e^{-\frac{2}{\pi a}} = e^{-\frac{2}{\pi}} \Rightarrow a = 1$$

15. (B)(C)

(A) $\lim_{x\to\infty}\frac{2x^2+1}{x^2+2} = \lim_{y\to 0^+}\frac{2+y^2}{1+2y^2}$

$$=\frac{1}{2}\lim_{y\to 0^+}\left(\frac{3}{1+2y^2}+1\right)=2^-$$

$$\Rightarrow\left[\lim_{x\to\infty}\frac{2x^2+1}{x^2+2}\right]=1$$

(B) $\lim_{x\to 0}\frac{\sin 2x}{x}=2\lim_{x\to 0}\frac{\sin 2x}{2x}=2^{-1}$

$$\Rightarrow\left[\lim_{x\to 0}\frac{\sin 2x}{x}\right]=1.$$

(C) $\lim_{x\to 0}\frac{\tan 3x}{x}=3\lim_{x\to 0}\frac{\tan 3x}{3x}=3^+$

$$\Rightarrow\left[\lim_{x\to 0}\frac{\tan 3x}{x}\right]=3.$$

(D) $\lim_{x\to -\infty}\frac{\sqrt{4x^2+1}}{x+2}=-\lim_{y\to 0^-}\frac{\sqrt{4+y^2}}{1+2y}=-\left(2^+\right)=-2^-$

$$\Rightarrow\left[\lim_{x\to -\infty}\frac{\sqrt{4x^2+1}}{x+2}\right]=-3$$

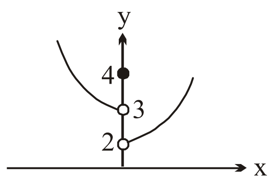
PASSAGE I

16. (C)

$x\to 0$, $x^3-x^2=x^2(x-1)\to 0^-$

$x\to 0$, $2x^4-x^5=x^4(2-x)\to 0^+$

$2(3)=\lambda\ (2)\ \Rightarrow\ \lambda=3$



17. (B)

$$\lim_{x\to 0^+}\frac{f(-x)x^2}{\left(\frac{1-\cos x}{[f(x)]}\right)-\left[\frac{1-\cos x}{[f(x)]}\right]}=\frac{3x^2}{\frac{1-\cos x}{2}-0}$$

$=6\times 2=12$

18. (B)

$x\to 0^-$ $\left(\frac{x^3-\sin^3 x}{x^4}\right)=\left(\frac{x-\sin x}{x^3}\right)\left(\frac{x^2+\sin^2 x+x\sin x}{x^2}\right)x=\frac{1}{6}\ (3)\ x\to 0^-\Rightarrow$

$f(0^-)=3$

$x\to 0^-$ $\frac{\sin x^3}{x}=\frac{\sin x^3}{x^3}\,x^2\to 0^+$ $\Rightarrow$ $\left[\frac{\sin x^3}{x}\right]=0$ $\Rightarrow$ $f(0)=4$

$\therefore$ $3f\left(\frac{x^3-\sin^3 x}{x^4}\right)>9$

$\Rightarrow$ $[9^+]-f(0)=9-4=5$

19. (A)

$$\left|\sum_{k=1}^{n}\left(3^k\left\{f(x+ky)-f(x-ky)\right\}\right)\right|\le 1....(i)$$

$$\Rightarrow\left|\sum_{k=1}^{n-1}\left(3^k\left\{f(x+ky)-f(x-ky)\right\}\right)\right|\le 1...(ii)$$

$$\Rightarrow\left|3^n\left\{f(x+ny)-f(x-ny)\right\}\right|\le 2$$

$$\Rightarrow\left|f(x+ny)-f(x-ny)\right|\le\frac{2}{3^n}$$

$$\Rightarrow\lim_{n\to\infty}\left|f(x+ny)-f(x-ny)\right|\le 0$$

$\Rightarrow f(x)$ is a cons tan t function .

20. (D)

$$\lim_{x\to 0}\frac{axe^x-b\ln(1+x)+cxe^{-x}}{x^2\sin x}=2$$

$$\Rightarrow\lim_{x\to 0}\frac{ax\left(1+x+\frac{x^2}{2}\right)-b\left(x-\frac{x^2}{2}+\frac{x^3}{3}\right)+cx\left(1-x+\frac{x^2}{2}\right)}{x^2\sin x}=2$$

$$\Rightarrow\lim_{x\to 0}\frac{(a-b+c)x+\left(a+\frac{b}{2}-c\right)x^2+\left(\frac{a}{2}-\frac{b}{3}+\frac{c}{2}\right)x^3}{x^3\frac{\sin x}{x}}=2$$

$$\Rightarrow a-b+c=0,\ a+\frac{b}{2}-c=0\ \&\ \frac{a}{2}-\frac{b}{3}+\frac{c}{2}=2$$

Hence $a=3, b=12, c=9$

Now g(x) = $3x^3 - 15x^2 + 9x$.

$$g(x)=f(x)\Rightarrow h(x)=3x^3-15x^2+9x+23=0$$

$$\Rightarrow h'(x)=9x^2-15x+9>0 \text{ for all } x\ .$$

Hence g(x) = f(x) has only one real root

21. (A)

$$\lim_{x\to 1}\frac{g(x)+3}{\ln(2-x)}=\lim_{x\to 1}\frac{3x^3-15x^2+9x+3}{\ln(2-x)}$$

By L'hospital's rule

$$\lim_{x\to 1}\frac{3x^3-15x^2+9x+3}{\ln(2-x)}=\lim_{x\to 1}\frac{9x^2-30x+9}{-\frac{1}{2-x}}=12$$

22. (A)

$$\lim_{x\to 0}\frac{\ln(f(x))}{\ln(g(x))}=\lim_{x\to 0}\frac{\lim\limits_{n\to\infty}\ln\left(\cos\sqrt{\frac{x}{n}}\right)^n}{\lim\limits_{n\to\infty}\ln\left(1+x\left(1+e^{1/n}\right)\right)^n}=\lim_{x\to 0}\lim_{n\to\infty}\frac{n\ln\left(\cos\sqrt{\frac{x}{n}}\right)}{n\ln\left(1+x\left(1+e^{1/n}\right)\right)}$$

$$=\lim_{x\to 0}\lim_{m\to 0}\frac{\ln\left(\cos\sqrt{mx}\right)}{\ln\left(1+x\left(1+e^{m}\right)\right)}=\lim_{x\to 0}\lim_{m\to 0}\frac{\frac{\ln\left(1-\left(1-\cos\sqrt{mx}\right)\right)}{1-\cos\sqrt{mx}}}{\frac{\ln\left(1+x\left(1+e^{m}\right)\right)}{x\left(1+e^{m}\right)}}\frac{1-\cos\sqrt{mx}}{x\left(1+e^{m}\right)}$$

$$=\lim_{x\to 0}\lim_{m\to 0}\frac{\frac{\ln\left(1-\left(1-\cos\sqrt{mx}\right)\right)}{1-\cos\sqrt{mx}}}{\frac{\ln\left(1+x\left(1+e^{m}\right)\right)}{x\left(1+e^{m}\right)}}\frac{2\sin^2\frac{\sqrt{mx}}{2}}{x\left(1+e^{m}\right)}$$

$$=\lim_{x\to 0}\lim_{m\to 0}\frac{\frac{\ln\left(1-\left(1-\cos\sqrt{mx}\right)\right)}{1-\cos\sqrt{mx}}}{\frac{\ln\left(1+x\left(1+e^{m}\right)\right)}{x\left(1+e^{m}\right)}}\frac{2\left(\frac{\sin\frac{\sqrt{mx}}{2}}{2\frac{\sqrt{mx}}{2}}\right)^2}{\left(\frac{1+e^{m}}{m}\right)}=\frac{1}{2}.$$

23. (C)

$$f(x)=e^{\lim\limits_{n\to\infty}n\left(\cos\sqrt{\frac{x}{n}}-1\right)}=e^{\lim\limits_{m\to 0}\frac{1}{m}\left(\cos\sqrt{mx}-1\right)}$$

$$f(x)=e^{\lim\limits_{m\to 0}\frac{-2\sin^2\frac{\sqrt{mx}}{2}}{m}}=e^{-2\lim\limits_{m\to 0}\left(\frac{\sin\frac{\sqrt{mx}}{2}}{\frac{\sqrt{mx}}{2}}\right)^2\frac{x}{4}}\Rightarrow f(x)=e^{-\frac{x}{2}}$$

$$g(x)=\lim_{n\to\infty}\left(1+x\left(1+e^{1/n}\right)\right)^n\quad\left\{1^\infty\ \text{ form}\right\}$$

$$g(x)=e^{\lim\limits_{n\to\infty}n\left(x\left(1+e^{1/n}\right)\right)}=e^{\lim\limits_{m\to 0}\left(x\left(\frac{1+e^{m}}{m}\right)\right)}\Rightarrow g(x)=e^{x}$$

$$f^{-1}(x)=-2\ln x,\ g^{-1}(x)=\ln x$$

$$\Rightarrow h(x)=\tan^{-1}\left(\ln\left(-2\ln x\right)\right)$$

Now $-2\ln x>0\Rightarrow \ln x<0\Rightarrow x\in(0,1)$

24. (D)

As range of $\ln(-2\ln x)$ is R, hence $-\frac{\pi}{2} < h(x) < \frac{\pi}{2}$.

25. (A)→(R); (B)→(S); (C)→(P); (D)→(Q)

(A) $l = \lim_{x\to0} \frac{e^{x^2} - 1 + x - e^x + 1}{2\frac{\sin^2 x}{x^2}\cdot x^2} = \frac{1}{2}\left[\lim_{x\to0}\frac{e^{x^2}-1}{x^2} + \lim_{x\to0}\frac{x-e^x+1}{x^2}\right] =$

$$\frac{1}{2}\left[1 - \lim_{x\to0}\frac{e^x - x - 1}{x^2}\right]$$

$$= \frac{1}{2}\left[1-\frac{1}{2}\right] = \frac{1}{4} \qquad \Rightarrow \qquad \frac{1}{l} = 4$$

(B) $l = \lim_{x\to0}\left(\frac{3+x}{3-x}\right)^{1/x} = e^{\lim_{x\to0}\frac{1}{x}\left(\frac{3+x}{3-x}-1\right)} = e^{\lim_{x\to0}\frac{2x}{x(3-x)}} = e^{2/3} \Rightarrow 2 + 3 = 5$

(C) $\lim_{x\to0}\frac{(\tan^3 x - x^3) - (\tan x^3 - x^3)}{x^5} = \lim_{x\to0}\frac{\tan^3 x - x^3}{x^5} - \underbrace{\lim_{x\to0}\frac{\tan x^3 - x^3}{x^5}}_{\text{zero (by expansion)}}$

$$= \lim_{x\to0}\frac{(\tan x - x)}{x^3}\cdot\frac{(\tan^2 x + x\tan x + x^2)}{x^2} = \frac{1}{3}\times 3 = 1$$

(D) rationalising gives

$$\lim_{x\to0}\frac{(x+2\sin x)\left[\sqrt{(x^2+2\sin x+1)}+\sqrt{\sin^2 x - x + 1}\right]}{(x^2+2\sin x+1)-(\sin^2 x - x + 1)}$$

$$2\cdot\lim_{x\to0}\frac{x+\sin 2x}{x^2-\sin^2 x+2\sin x+x} = 2\cdot\lim_{x\to0}\frac{1+\frac{\sin 2x}{x}}{x-\frac{\sin^2 x}{x}+2+1} = 2\left(\frac{1+2}{3}\right) = 2.$$

26. (A)→(Q), (B)→(R), (C)→(P), (D)→(P)

(A) $\lim_{n\to\infty}\cos^2\left(\pi\left(\sqrt[3]{n^3+n^2+2n}\right)\right) = \lim_{n\to\infty}\cos^2\left(n\pi - \left(n^3+n^2+2n\right)^{1/3}\pi\right)$

$$= \lim_{n\to\infty}\cos^2\pi\left(\frac{n^2+2n}{n^2+n\left(n^3+n^2+2n\right)^{1/3}+\left(n^3+n^2+2n\right)^{2/3}}\right)$$

$$= \lim_{n\to\infty}\cos^2\pi\left(\frac{1+\frac{2}{n}}{1+\left(1+\frac{1}{n}+\frac{2}{n^2}\right)^{1/3}+\left(1+\frac{1}{n}+\frac{2}{n^2}\right)^{2/3}}\right) = \cos^2\frac{\pi}{3} = \frac{1}{4}.$$

(B) $\lim_{n\to\infty} \dfrac{\sin\left(2\pi\sqrt{1+n^2}\right)}{1/n} \quad \to \left\{\dfrac{0}{0} \text{ form as } \sin\left(2\pi\sqrt{1+n^2}\right)\to 0\right\}$

Also $\lim_{n\to\infty}\left(n-\sqrt{1+n^2}\right) = \lim_{n\to\infty}\left(\dfrac{-1}{n+\sqrt{1+n^2}}\right) = 0$

$$\therefore \lim_{n\to\infty} \frac{\sin\left(2\pi\sqrt{1+n^2}\right)}{1/n} = -\lim_{n\to\infty} \frac{\sin\left(2n\pi - 2\pi\sqrt{1+n^2}\right)}{2n\pi - 2\pi\sqrt{1+n^2}} 2n\pi\left(n-\sqrt{1+n^2}\right)$$

$$= \lim_{n\to\infty} \frac{\sin\left(2n\pi - 2\pi\sqrt{1+n^2}\right)}{2n\pi - 2\pi\sqrt{1+n^2}} \frac{2n\pi}{n+\sqrt{1+n^2}}$$

$$= \lim_{n\to\infty} \frac{\sin\left(2n\pi - 2\pi\sqrt{1+n^2}\right)}{2n\pi - 2\pi\sqrt{1+n^2}} \frac{2\pi}{1+\sqrt{\frac{1}{n^2}+1}} = \pi$$

(C) $\operatorname{Lim}_{n\to\infty} (-1)^n \sin\left(\pi\sqrt{n^2+0.5n+1}\right)$

$$= -\operatorname{Lim}_{n\to\infty} \pi\left(n-\sqrt{n^2+0.5n+1}\right)\frac{\sin\left(n\pi - \pi\sqrt{n^2+0.5n+1}\right)}{\pi n - \pi\sqrt{n^2+0.5n+1}}$$

$$= \operatorname{Lim}_{n\to\infty} \left(\frac{0.5n+1}{n+\sqrt{n^2+0.5n+1}}\right)\frac{\sin\left(\pi - \pi\sqrt{n^2+0.5n+1}\right)}{\pi - \pi\sqrt{n^2+0.5n+1}}$$

$$= \operatorname{Lim}_{n\to\infty} \left(\frac{\frac{1}{2}+\frac{1}{n}}{1+\sqrt{1+\frac{1}{2n}+\frac{1}{n^2}}}\right)\frac{\sin\left(\pi - \pi\sqrt{n^2+0.5n+1}\right)}{\pi - \pi\sqrt{n^2+0.5n+1}} = \frac{1}{2}.$$

(D) $\lim_{x\to\infty}\left(\dfrac{x+a}{x-a}\right)^x = e^{\lim_{x\to\infty} x\left(\frac{x+a}{x-a}-1\right)} = e^{\lim_{x\to\infty}\left(\frac{2ax}{x-a}\right)} = e^{2a} \Rightarrow a = \dfrac{1}{2}$

27. (A)$\to$(S), (B)$\to$(P), (C)$\to$(Q), (D)$\to$(R)

(A) $\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}\right) = \lim_{x\to\infty}\left(\dfrac{2\sqrt{x}}{\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}}\right)$

$$= \lim_{x\to\infty}\left(\frac{2}{\sqrt{1+\frac{1}{\sqrt{x}}}+\sqrt{1-\frac{1}{\sqrt{x}}}}\right) = 1$$

(B) $\lim\limits_{x\to 0}\dfrac{\sin 2x-2\tan x}{\ln(1+x^3)}=\lim\limits_{x\to 0}\dfrac{2\sin x\cos x-2\dfrac{\sin x}{\cos x}}{\ln(1+x^3)}$

$=-\lim\limits_{x\to 0}\dfrac{2\sin^3 x}{\cos x\ln(1+x^3)}=-2\lim\limits_{x\to 0}\dfrac{\dfrac{\sin^3 x}{x^3}}{\cos x\dfrac{\ln\left(1+x^3\right)}{x^3}}=-2$.

(C) $\lim\limits_{x\to 0^+}\left(\ln\sin^3 x-\ln\left(x^4+ex^3\right)\right)=\lim\limits_{x\to 0^+}\left(3\ln\sin x-3\ln x-\ln\left(x+e\right)\right)$

$=\lim\limits_{x\to 0^+}\left(3\ln\dfrac{\sin x}{x}-\ln\left(x+e\right)\right)=-1$

(D) $\tan\left(2\pi|\sin\theta|\right)=\cot\left(2\pi|\cos\theta|\right)\Rightarrow\tan\left(2\pi|\sin\theta|\right)=\tan\left(\dfrac{\pi}{2}-2\pi|\cos\theta|\right)$

$\Rightarrow 4|\sin\theta|=2n+1-4|\cos\theta|$

$\Rightarrow|\sin\theta|+|\cos\theta|=\dfrac{2n+1}{4}$.

Now range of $|\sin\theta|+|\cos\theta|$ is $\left[1,\sqrt{2}\right]$ hence $\dfrac{2n+1}{4}=\dfrac{5}{4}$

Further $f\left(x\right)=\left(\dfrac{5}{4}\right)^x$

$\Rightarrow\lim\limits_{x\to\infty}\left[\dfrac{2}{f\left(x\right)}\right]=\lim\limits_{x\to\infty}\left[2\left(\dfrac{4}{5}\right)^x\right]=0$.

28. (A)$\to$(S); (B)$\to$(R); (C)$\to$(P); (D)$\to$(Q); (E)$\to$(P)

(A) $\lim\limits_{x\to 1}\dfrac{\ln x}{x^4-1}=\lim\limits_{x\to 1}\dfrac{\ln x}{x-1}\dfrac{1}{\left(x^2+1\right)\left(x+1\right)}=\dfrac{1}{4}$

(B) $\lim\limits_{x\to 0}\dfrac{3e^x-x^3-3x-3}{\tan^2 x}=\lim\limits_{x\to 0}\dfrac{3\left(1+x+\dfrac{x^2}{2}\right)-x^3-3x-3}{\tan^2 x}$

$=\lim\limits_{x\to 0}\dfrac{\dfrac{3x^2}{2}-x^3}{\tan^2 x}=\lim\limits_{x\to 0}\dfrac{\dfrac{3}{2}-x}{\dfrac{\tan^2 x}{x^2}}=\dfrac{3}{2}$

(C) $\lim\limits_{x\to\infty}\dfrac{\pi-2\tan^{-1}x}{\ln\left(1+\dfrac{1}{x}\right)}=2\lim\limits_{x\to\infty}\dfrac{\cot^{-1}x}{\ln\left(1+\dfrac{1}{x}\right)}$

$$= 2\lim_{x\to\infty} \frac{\dfrac{\tan^{-1}\frac{1}{x}}{\frac{1}{x}}}{\dfrac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}} = 2$$

(D) $$\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x(\cos x - \cos 2x)} = -\lim_{x\to 0} \frac{2\sin x(1-\cos x)}{x(2\cos^2 x - \cos x - 1)}$$

$$= -\lim_{x\to 0} \frac{2\sin x(1-\cos x)}{x(2\cos x + 1)(\cos x - 1)} = 2$$

(E) $$\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} = \lim_{x\to 0} \frac{\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right)-\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}\right)-2x}{x-\left(x-\frac{x^3}{6}\right)}$$

$$= \lim_{x\to 0} \frac{\frac{x^3}{3}}{\frac{x^3}{6}} = 2.$$

29. (A)$\to$(S); (B)$\to$(R); (C)$\to$(Q); (D)$\to$(P)

(A) $$\lim_{x\to\infty} \left(\frac{x}{1+x}\right)^x = e^{\lim_{x\to\infty} x\left(\frac{x}{1+x}-1\right)} = e^{\lim_{x\to\infty}\left(\frac{-x}{1+x}\right)} = \frac{1}{e}$$

(B) $$\lim_{x\to\infty} \left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^x = e^{\lim_{x\to\infty} x\left(\sin\frac{1}{x}+\cos\frac{1}{x}-1\right)} = e^{\lim_{x\to\infty}\left(\frac{\sin\frac{1}{x}}{\frac{1}{x}}+\frac{\cos\frac{1}{x}-1}{\frac{1}{x}}\right)}$$

$$= e^{\lim_{x\to\infty}\left(\frac{\sin\frac{1}{x}}{\frac{1}{x}}-\left(\frac{2\sin^2\frac{1}{2x}}{\frac{1}{4x^2}}\right)\frac{1}{4x}\right)} = e$$

(C) $$\lim_{x\to 0} (\cos x)^{\cot^2 x} = e^{\lim_{x\to 0} \cot^2 x(\cos x - 1)} = e^{-\lim_{x\to 0}\left(\frac{2\sin^2\frac{x}{2}}{\tan^2 x}\right)}$$

$$= e^{-\frac{1}{2}\lim_{x\to 0}\left(\frac{\sin^2\frac{x}{2}}{\frac{x^2}{4}}\cdot\frac{x^2}{\tan^2 x}\right)} = e^{-\frac{1}{2}}$$

(D) $$\lim_{x\to 0}\left(\tan\left(\frac{\pi}{4}+x\right)\right)^{1/x} = e^{\lim_{x\to 0}\frac{1}{x}\left(\tan\left(\frac{\pi}{4}+x\right)-1\right)} = e^{\lim_{x\to 0}\frac{1}{x}\left(\frac{1+\tan x}{1-\tan x}-1\right)} = e^{\lim_{x\to 0}\frac{2}{1-\tan x}\left(\frac{\tan x}{x}\right)} = e^2.$$