# HYPERBOLA

## EXERCISE – 3

**Q.1**

Let middle point is P(h , k).

The equation of chord is T = S’

$xh - yk = h^2 - k^2$

Let this is normal at $P(a\sec\theta\ ,\ a\tan\theta)$. So equation of normal is $\frac{x}{\sec\theta} + \frac{y}{\tan\theta} = 2a$.

On comparison : $h\sec\theta = -k\tan\theta = \frac{h^2 - k^2}{2a}$

Eliminating $'\theta'$; we get $\left(\frac{h^2 - k^2}{2a}\right)^2 \left[\frac{1}{h^2} - \frac{1}{k^2}\right] = 1$

$\Rightarrow (x^2 - y^2)^3 = 4a^2x^2y^2$

**Q.2**

Let vertices of triangle are $A(t_1)\ ,\ B(t_2)\ ,\ C(t_3)$

$AB : (t_1t_2)y + x = C(t_1 + t_2)$

$BC : (t_2 + t_3)y + x = C(t_1 + t_3)$

$AC : (t_1 + 3)y + x = C(t_1 + t_3)$

If $(t_1t_2)y + x - c(t_1 + t_2) = 0$ is tangent to $y^2 = 4ax$. Then $\frac{y^2}{4a} + (t_1t_2)y - C(t_1 + t_2) = 0$ has discriminate equal to zero.

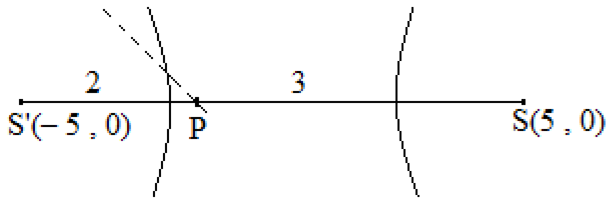
So D = 0 $\Rightarrow$ $a(t_1t_2)^2 + Ct_1 + Ct_2 = 0$

Similarly $a(t_2t_3)^2 + Ct_2 + Ct_3 = 0$

$a(t_1t_3)^2 + Ct_1 + Ct_3 = 0$

There are infinitely possible solutions for $t_1\ ,\ t_2\ \&\ t_3$

**Q.3**




$$\frac{x^2}{16} - \frac{y^2}{5} = 1$$

$$e = \sqrt{1 + \frac{5}{16}}$$

$$e = \frac{5}{4}$$

P is (– 1 , 0)

Equation of line is ( y – 0 ) = – 1 ( x + 1 )

$\Rightarrow y + x + 1 = 0$

Asymptotes of hyperbola are $9x^2 - 16y^2 = 0$

$\Rightarrow 9x^2 - 16(x+1)^2 = 0$

$\Rightarrow (3x - 4x - 4)\ (3x + 4x + 4) = 0$

$\Rightarrow x = -4 \quad \& \quad x = \frac{-4}{7}$

$\Rightarrow y = 3 \quad \& \quad y = \frac{-3}{7}$

So points are ( – 4 , 3 ) & $\left(\frac{-4}{7}, \frac{-3}{7}\right)$

**Q.4**

Asymptote of $\frac{x^2}{16} - \frac{y^2}{9} = 1$ are $y = \pm \frac{3}{4}x$

Diameters of the ellipse perpendicular to this asymptotes are $y = \pm \frac{4}{3}x$

Passing through I[st] & III[rd] is $y = \frac{4}{3}x$ .

Length of diameter of slope m $= 2ab\sqrt{\frac{1+m^2}{b^2+a^2m^2}}$

Hence required length is $= \frac{150}{\sqrt{481}}$.

**Q.5**

Let P is $(a\sec\theta \,,\, b\tan\theta)$.

Tangent at P is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

Let it meet $y = \frac{b}{a}x$ at $\theta$.

So $x = \frac{a}{\sec\theta - \tan\theta} = a(\sec\theta + \tan\theta)$

$y = b(\sec\theta + \tan\theta)$

Mid – point of PQ be (h , k)

$h = a\left(\sec\theta + \frac{\tan\theta}{2}\right) \;;\; k = g\left(\frac{\sec\theta}{2} + \tan\theta\right)$

Eliminating $\theta \Rightarrow 4\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = 3$

**Q.6**

$x\cos\alpha + y\sin\alpha = p$ is tangent to $x^2 + y^2 = p^2$

**Q.7**

Chord is : $\frac{x\cos\theta - \varphi}{2} - \frac{y\sin\theta + \varphi}{2} = \frac{\cos\theta + \varphi}{2}$

Normal at P : $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$

So $\frac{\cos\frac{\theta-\varphi}{2}}{} \cdot \sec\theta = \frac{-\sin(\theta+\varphi)}{2}\tan\theta$

$$= \frac{\overline{\cos\theta+\varphi}}{\frac{2}{a^2+b^2}}$$

On simplifying we get $\boxed{\tan\varphi = \tan\theta(4\sec^2\theta - 1)}$

**Q.8**

Let middle point is (h , k)

Chord is : $\frac{xh}{a^2} - \frac{yk}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$

Chord of contact from $(r\cos\theta\ ,\ r\sin\theta)$ is $\frac{x\,r\cos\theta}{a^2} - \frac{y\,r\sin\theta}{b^2} = 1$

$$\frac{h}{r\cos\theta} = \frac{k}{r\sin\theta} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\Rightarrow \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 = \frac{x^2+y^2}{r^2}$$

**Q.9**

Equation of pair of asymptotes of hyperbola differ from the equation of the hyperbola by a constant. Let the equation of pair of asymptotes be

$2x^2 - 3xy - 2y^2 + 3x - y + \lambda = 0$. It passes through the centre of the hyperbola.

$$\left.\begin{aligned} \frac{ds}{dx} &= 4x - 3y + 3 = 0 \\ \frac{ds}{dy} &= -3x - 4y - 1 = 0 \end{aligned}\right\} \quad \text{solving we get } \left(y = \frac{1}{5}\ ,\ x = -\frac{12}{5}\right)$$

Asymptotes pass through $\left(-\frac{12}{5}, \frac{1}{5}\right)$

$\boxed{\lambda = -6}$

**Q.10**

Let asymptotes are $(2x + 3y + \lambda_1) = 0$ and $(3x+2y+\lambda_2) = 0$. Asymptote pass through (1, 2). So, $\lambda_1 = -8$ , $\lambda_2 = -7$.

Hyperbola is $(2x+3y-8)\ (3x+2y-7) + \lambda = 0$. It passes through (5, 3).

So we get $(11)\ (14) + \lambda = 0 \ \Rightarrow \ \lambda = -154$.

So hyperbola is $6x^2 + 13xy + 6y^2 - 38x - 37y - 98 = 0$

**Q.11**

Let P is $(a\sec\theta\ , b\tan\theta)$. Let $\tan\alpha = m$ A point PQR at a distance 'r' from P is $(a\sec\theta + r\cos\alpha\ ,\ b\tan\theta + r\sin\alpha)$. It lies on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$. So,

$b^2(a\sec\theta + r\cos\alpha)^2 = a^2(b\tan\theta + r\sin\alpha)^2$

$$\Rightarrow\ r^2\left[b^2\cos^2\alpha - a^2\sin^2\alpha\right] + 2r(b^2a\sec\theta\cos\alpha - a^2b\tan\theta\sin\alpha) + b^2a^2\sec^2\theta - a^2b^2\tan^2\theta$$

$$\Rightarrow r_1r_2 = PQ \cdot PR = \frac{b^2a^2(\sec^2\theta - \tan^2\theta)}{(b^2\cos^2\alpha - a^2\sin^2\alpha)}$$

$$(QP)\cdot(PR) = \frac{b^2a^2}{b^2\cos^2\alpha - a^2\sin^2\alpha}$$

$$\tan\alpha = m \quad \Rightarrow\ \sin^2\alpha = \frac{m^2}{1+m^2}$$

$$\Rightarrow\ \cos^2\alpha = \frac{1}{1+m^2}$$

So, $(PQ)(PR) = \dfrac{b^2a^2(1+m^2)}{b^2 - a^2m^2}$

**Q.12**

Equation of tangents from (3 , 2) $y = mx + \sqrt{9m^2 - 1}$. It goes through (3 , 2).

So, $(2-3m)^2 = (9m^2 - 1)$

$\Rightarrow\ 4+9m^2-12m=9m^2-1$

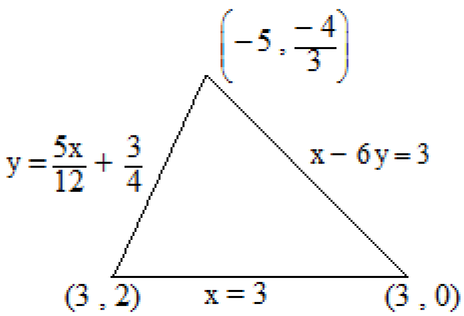
$\Rightarrow\ 12m = 5$

$\Rightarrow\ m = \frac{5}{12}$ other root is $\infty$.

So tangent are $\boxed{y = \frac{5x}{12} + \frac{3}{4}}$ & $\boxed{x = 3}$

Chord of contact is 3x – 9y(2) = 9 i.e. $\boxed{x-6y=3}$

Area of triangle can now be obtained




Which is 8 sq. units.

**Q.13**

Let the chords be $y=m(x-ae)$ & $y=-\frac{1}{m}(x+ae)$

Eliminating m gives $y^2=-(x+ae)(x-ae)$ or $x^2+y^2=a^2e^2$ as the required locus.

**Q.14**

$x=t^2+t+1, y=t^2-t+1 \Rightarrow x-y=2t$

$\Rightarrow x=\left(\frac{x-y}{2}\right)^2+\frac{x-y}{2}+1$

$\Rightarrow \frac{x+y-2}{2}=\left(\frac{x-y}{2}\right)^2$

The required locus is in standard form of equation of parabola.

**Q.15**

Let the mid-point be (h, k), then equation of chord $\left(T = S_1\right)$ be $hx - ky = a^2$.

Also equation of any tangent to $y^2 = 4ax$ be $ty = x + at^2$.

Comparing the two equations gives, $\frac{k}{t} = h = -\frac{a}{t^2}$ or $k^2 = -ah$

Hence required locus is $y^2 = -ax$.

**Q.16**

Tangent to the hyperbola at $P(\theta)$ is $\frac{x}{2}\sec\theta - \frac{y}{3}\tan\theta = 1$.

Comparing this with $3x - y = c$ gives

$\frac{\sec\theta}{6} = \frac{\tan\theta}{3} \Rightarrow \sin\theta = \frac{1}{2}$ or $\theta = \frac{\pi}{6}$.

**Q.17**

Let the common tangent be $y = mx + c$, then

for $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,\ c^2 = a^2m^2 - b^2$ & for $\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1,\ c^2 = a^2 - b^2m^2$.

Hence $a^2m^2 - b^2 = a^2 - b^2m^2$ or $m = \pm 1$.

Hence the common tangents are $y = \pm x \pm \sqrt{a^2 - b^2}$.

**Q.18**

Let any point on $S_1$ be $\left(a\sec\theta, b\tan\theta\right)$.

Chord of contact of $S_2$ w.r.to this point will be $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 2$ ...(i)

Also asymptotes of $S_1$ are $\frac{x}{a} \pm \frac{y}{b} = 0$.

Solving these with (i) gives points of intersections as
$\left(2a\left(\sec\theta + \tan\theta\right), 2b\left(\sec\theta + \tan\theta\right)\right)$ & $\left(2a\left(\sec\theta - \tan\theta\right), -2b\left(\sec\theta - \tan\theta\right)\right)$

Now area of triangle formed by these and the origin

$$= \frac{1}{2}\left\|\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2a(\sec\theta + \tan\theta) & 2b(\sec\theta + \tan\theta) \\ 1 & 2a(\sec\theta - \tan\theta) & -2b(\sec\theta - \tan\theta) \end{vmatrix}\right\| = 4ab.$$

**Q.19**

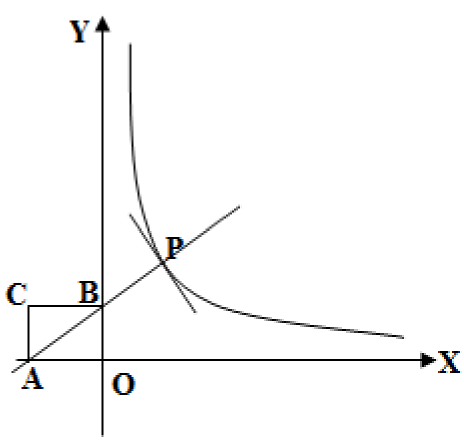
$$(10x-5)^2 + (10y-2)^2 = 9(3x+4y-7)^2 \Rightarrow \left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{5}\right)^2 = \frac{9}{4}\left(\frac{3x+4y-7}{5}\right)^2$$

Hence one focus is $\left(\frac{1}{2},\frac{1}{5}\right)$, corresponding directrics is 3x + 4y = 7 and eccentricity is $\frac{3}{2}$.

So the latus rectum will be parallel to directrics passing through the focus

i.e. 3x + 4y = $\frac{23}{10}$.

**Q.20**




Let P be $\left(t,\frac{1}{t}\right)$, then normal at P will be $t^3x - ty = t^4 - 1$.

Hence coordinates of A & B will be $\left(\frac{t^4-1}{t^3},0\right)$ & $\left(0,-\frac{t^4-1}{t}\right)$ and

coordinates of P will be $x = \frac{t^4-1}{t^3}, y = -\frac{t^4-1}{t}$.

Eliminating 't' gives the required locus as $\left(x^2 - y^2\right)^2 + x^3y^3 = 0$.

**Q.21**

Equation of chord joining $P(\theta_1) \& Q(\theta_2)$ will be

$$\frac{x}{a}\cos\frac{\theta_1-\theta_2}{2}-\frac{y}{b}\sin\frac{\theta_1+\theta_2}{2}=\cos\frac{\theta_1+\theta_2}{2}$$

If it passes through ( $\pm$ ae, 0), then $\dfrac{\cos\frac{\theta_1-\theta_2}{2}}{\cos\frac{\theta_1+\theta_2}{2}}=\dfrac{1}{\pm e}$.

$$\Rightarrow \frac{\cos\frac{\theta_1-\theta_2}{2}-\cos\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}+\cos\frac{\theta_1+\theta_2}{2}}=\frac{1\mp e}{1\pm e} \text{ or } \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}=\frac{1\mp e}{1\pm e}.$$

**Q.22**

Locus of P, such that $|PA - PB| = k$, is a hyperbola if $0 < k < AB$.

Hence k must be less than the distance between (0, -1) & (0, 1)

i.e. $0 < k < 2$.

**Q.23**

Let P be $\left(t, \frac{1}{t}\right)$, then tangent and normal at P will be $x+t^2y=2ct$ & $t^3x-ty=t^4-1$.

Now $a_1=2ct, b_1=\dfrac{2c}{t}, a_2=\dfrac{c(t^4-1)}{t^3}$ & $b_2=-\dfrac{c(t^4-1)}{t}$

$$\Rightarrow a_1a_2+b_1b_2=2ct\times\frac{c(t^4-1)}{t^3}-\frac{2c}{t}\times\frac{c(t^4-1)}{t}=0.$$

**Q.24**

Let P be $\left(ct_1, \frac{c}{t_1}\right)$, then normal at P will be $t_1^3x-t_1y=c(t_1^4-1)$.

If this normal meets the curve again at $\left(ct_2, \frac{c}{t_2}\right)$, then

$$t_1^3 t_2 - \frac{t_1}{t_2} = t_1^4 - 1$$

$$\Rightarrow t_1^3 t_2 \left(t_2 - t_1\right) = t_1 - t_2$$

$$\Rightarrow t_1^3 t_2 = -1.$$