HYPERBOLA

EXERCISE – 2(C)

Q.1

$$\frac{2b^2}{a} = 2b \qquad \Rightarrow \frac{b}{a} = 1;$$

So slopes of asymptotes are ± 1.

$\therefore$ asymptotes perpendicular

Q.2

Let the hyperbolas be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

A line parallel to y – axis, say x = k, meets these in P & Q.

Now $\frac{k^2}{a^2} - \frac{y^2}{b^2} = 1$ & $x = k \Rightarrow y = b\sqrt{\frac{k^2}{a^2} - 1}$, hence P is $\left(k, b\sqrt{\frac{k^2}{a^2} - 1}\right)$

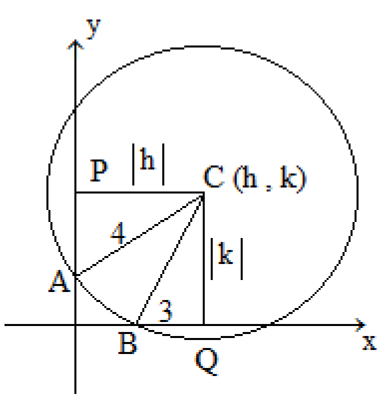
and $\frac{k^2}{a^2} - \frac{y^2}{b^2} = -1$ & $x = k \Rightarrow y = b\sqrt{\frac{k^2}{a^2} + 1}$, hence Q is $\left(k, b\sqrt{\frac{k^2}{a^2} + 1}\right)$

Normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at P will be $\left(\frac{x-k}{k}\right)a^2 + \left(\frac{ay - b\sqrt{k^2 - a^2}}{b\sqrt{k^2 - a^2}}\right)b^2 = 0$ …(1)

Normal to $\frac{k^2}{a^2} - \frac{y^2}{b^2} = -1$ at Q will be $\left(\frac{x-k}{k}\right)a^2 + \left(\frac{ay - b\sqrt{k^2 + a^2}}{b\sqrt{k^2 + a^2}}\right)b^2 = 0$ …(2)

Putting y = 0 in (1) & (2) gives $x = \frac{b^2 k}{a^2} + k$, hence the point of intersection lies on x – axis.

Q.3



CA = CB $\Rightarrow h^2+16=k^2+9$

$\Rightarrow \boxed{y^2-x^2=7}$

Now foci of $x^2-y^2=-a^2$ are $\left(0,\pm\sqrt{2}a\right)$, hence the foci are $\left(0,\pm\sqrt{14}\right)$.

Q.4

Let P be $\left(a\sec\theta, b\tan\theta\right)$.

Normal at P : $\dfrac{ax}{\sec\theta}+\dfrac{by}{\tan\theta}=a^2+b^2$

It meets transverse axis(x – axis) at $G\left(\dfrac{a^2+b^2}{a}\sec\theta, 0\right)$

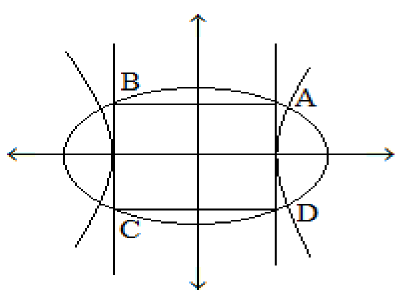
Slope of one of the asymptotes = $\dfrac{b}{a}$

Now GL will be $y=-\dfrac{a}{b}\left(x-\dfrac{a^2+b^2}{a}\sec\theta\right)$

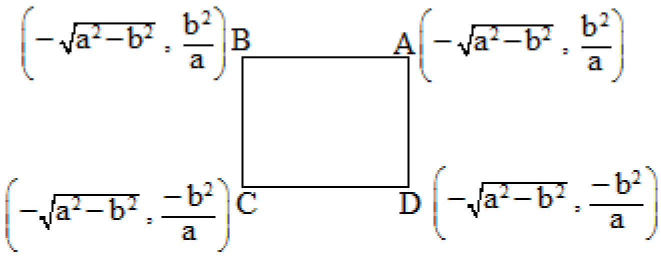
It will meet the asymptote $bx=ay$ at $L\left(a\sec\theta, b\sec\theta\right)$

Clearly slope of LP = 0.

Q.5



AD : $x=\sqrt{a^2-b^2}$ & CB : $x=-\sqrt{a^2-b^2}$

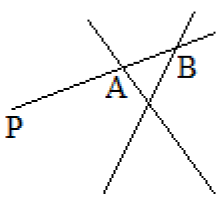


So area of rectangle ABCD is $= \left(2\sqrt{a^2-b^2}\right)\left(\frac{2b^2}{a}\right)$

$$= \frac{4b^2\sqrt{a^2-b^2}}{a}$$

Q.6

Let ‘P’ is $\left(\frac{\sec\theta}{b}, \frac{\tan\theta}{a}\right)$. Combined equation of asymptotes is $b^2x^2 - a^2y^2 = 0$



By parametric from a point on line PA will be

$$\left(\frac{\sec\theta}{b} + r\cos\alpha, \frac{\tan\theta}{a} + r\sin\alpha\right)$$

It lies on asymptotes then

$$b^2\left(\frac{\sec\theta}{b} + r\cos\alpha\right)^2 - a^2\left(\frac{\tan\theta}{a} + r\sin\alpha\right)^2 = 0$$

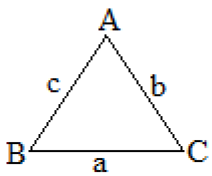
$$\Rightarrow r^2(b^2\cos^2\alpha - a^2\sin^2\alpha) + 2(b\sec\theta\cos\alpha - a\tan\theta\sin\alpha)r + (\sec^2\theta - \tan^2\theta) = 0$$

So $PA\cdot PB = r_1\cdot r_2 = \frac{\sec^2\theta - \tan^2\theta}{b^2\cos^2\alpha - a^2\sin^2\alpha}$

$PA\cdot PB = \frac{1}{b^2\cos^2\alpha - a^2\sin^2\alpha}$ (independent of θ)

Hence $PA \cdot PB$ is independent of point P.

Q.7



BC = a is fixed

$$\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} \quad \& \quad \tan\frac{c}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}} = \frac{s-c}{s-b} = k \Rightarrow \frac{a+b-c}{a+c-b} = k$$

$$\Rightarrow a+b-c = ka+kc-kb$$

$$\Rightarrow (k+1)c - (k+1)b = (1-k)a$$

$$\Rightarrow c-b = \left(\frac{1-k}{1+k}\right) a = a \text{ constant}$$

$\Rightarrow$ BA – CA is constant.

So locus of A is a hyperbola with B and C as foci.

Q.8

Let any tangent to $y^2 = 4ax$ with slope m be $y = mx + \frac{a}{m}$.

Now if two tangents are drawn from P(h, k), then slopes of these tangents $(m_1 \& m_2)$ will be the roots of $k = mh + \frac{a}{m}$ or $hm^2 - km + a = 0$.

$$\therefore m_1 + m_2 = \frac{k}{h} \;\&\; m_1 m_2 = \frac{a}{h}$$

$$\text{Now } \tan 45^o = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right| \Rightarrow (m_1 - m_2)^2 = (1+m_1 m_2)^2$$

$$\Rightarrow (m_1 + m_2)^2 - 4m_1 m_2 = (1+m_1 m_2)^2$$

$\Rightarrow \frac{k^2}{h^2} - \frac{4a}{h} = \left(1 + \frac{a}{h}\right)^2$

$\Rightarrow k^2 - 4ah = h^2 + 2ah + a^2$

$\Rightarrow (h + 3a)^2 - k^2 = 8a^2$

Hence required locus is a hyperbola.

Q.10

Factorizing $x^2 + 2xy - 3y^2 = 0$ gives x + 3y & x – y as factors.

Now let the asymptotes be x + 3y + a = 0 & x – y + b = 0

Pair of asymptotes will be $x^2 + 2xy - 3y^2 + (a + b)x + (3b - a)y + ab = 0$

Comparing it with $x^2 + 2xy - 3y^2 + x + 7y + c = 0$ gives

a + b = 1, 3b – a = 7 & ab = c.

$\Rightarrow a = -1, b = 2, c = -2$.

Hence the asymptotes are x + 3y – 1 = 0 & x – y + 2 = 0.

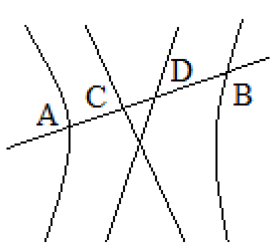
$m_1 = -3, m_2 = 1 \Rightarrow \tan\theta = \frac{-3-1}{1-1\times 3} = 2$

$\therefore$ angle between the asymptotes $= \tan^{-1} 2$.

Q.11

Let the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Let the line is y= mx + c



Solving y = mx + c & $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

We have $\frac{x^2}{a^2} - \frac{(mx+c)}{b^2} = 1$

$$\Rightarrow \left(\frac{1}{a^2} - \frac{m^2}{b^2}\right)x^2 - \left(\frac{2mc}{b^2}\right)x - \left(1+\frac{c^2}{b^2}\right) = 0$$

Let A, B are (x_1, y_1) & (x_2, y_2) mid – point of AB is $\left(\frac{x_1+x_2}{2}, \frac{m(x_1+x_2)}{2} + c\right)$

$x_1 + x_2$ is sum of roots of above quadratic.

Equation of asymptotes are $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

On solving we have the quadratic

$$\left(\frac{1}{a^2} - \frac{m^2}{c^2}\right)x^2 - \left(\frac{2mc}{b^2}\right)x - \frac{c^2}{b^2} = 0$$

So again $x_1 + x_2$ is same, hence mid – pint is same.

Q.12

Normal at any point $\left(t, \frac{1}{t}\right)$ is $(ty-1) = t^3(x-t)$. It passes through (h , k).

So, $(tk-1) = t^3(h-t) \Rightarrow t^4 - ht^3 + tk - 1 = 0$

We have four roots of above equation. Let variable line is ax + by + c = 0

We have $\frac{a(x_1+x_2+x_3+x_4) + b(y_1+y_2+y_3+y_4) + 4c}{\sqrt{a^2+b^2}} = c$

$$\sum x = \sum t, \ \sum y = \sum \frac{1}{t}$$

$$\sum t = h; \ \sum \frac{1}{t} = \frac{\sum t_1t_2t_3}{t_1t_2t_3t_4} = \frac{-k}{-1} = k$$

$$\Rightarrow ah + bk + 4c = 0$$

$\Rightarrow a\left(\frac{h}{4}\right)+b\left(\frac{k}{4}\right)+c=0$

So line passé through $\left(\frac{h}{4},\frac{h}{4}\right)$

Q.13

Conjugate hyperbola : $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$

Let P is $(a\tan\theta, b\sec\theta)$.

Equation of tangent is $\boxed{\frac{y\sec\theta}{b}-\frac{x\tan\theta}{a}=1}$

And chord to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with P as mid – point is T = S'

$\Rightarrow \frac{x\tan\theta}{a}-\frac{y\sec\theta}{b}=\tan^2\theta-\sec^2\theta$

$\Rightarrow \frac{a\tan\theta}{a}-\frac{y\sec\theta}{a}=-1$

$\Rightarrow \boxed{\frac{y\sec\theta}{b}-\frac{x\tan\theta}{a}=1}$ which is same as above equation.

Q.14

A chord joining $A(t_1)$ & $B(t_2)$ on the curve $xy=c^2$ subtends right angle at $P(t_3)$

So we have slope of $AP=\frac{-1}{t_1+3}$ & slope of $BP=\frac{-1}{t_2t_3}$

$\Rightarrow \frac{1}{t_1t_2t_3 2}=-1$ as $\angle APB=90^\circ$

slope of AB : $\frac{-1}{t_1t_2}$

slope of normal at P : $t_3{}^2$

we have : $t_3^{\,2} = \dfrac{-1}{t_1 t_2}$

Q.15

Polar of $(x_1\ ,\ y_1)$ with respect to $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$. This line is tangent to $xy = c^2$.

On sharing we have $(x_1)x + \dfrac{(c^2 y_1)}{x} = a^2$

$\Rightarrow (x_1)x^2 - (a^2)x + (c^2 y_1) = 0$

$D = a^4 - 4x_1 y_1 c^2 = 0$

$\Rightarrow x_1 y_1 = \dfrac{a^4}{4c^2}$

So $(x_1\ ,\ y_1)$ lies on $xy = \dfrac{a^4}{c^4}$ (A concentric rectangular hyperbola to $xy = c^2$)

Q.16

$xy = c^2 \Rightarrow \dfrac{dy}{dx} = \dfrac{-c^2}{x^2}$

$\Rightarrow \dfrac{-dx}{dy} = \dfrac{x^2}{c^2}$

Slope of normal at $P(t_1)$ is t_1^2. Equation of normal is $\left(y - \dfrac{c}{t_1}\right) = t_1^2\,(x - c\,t_1)$

$\Rightarrow (t_1 y - c) = t_1^3 (x - c\,t_1)$

$\Rightarrow (t_1^3)x - (t_1)y + c(1 - t_1^4) = 0$

Let is meets curve at $\left(ct\ ,\ \dfrac{c}{t}\right)$.So

$\Rightarrow (c\,t_1^3)t - (t_1)\dfrac{c}{t} + c(1 - t_1^4) = 0$

$\Rightarrow t^2 (t_1^3) + t(1 - t_1^4) - (t_1) = 0$

The two roots are t_1 & t_2. So

$$t_1 t_2 = \frac{-t}{t_1^3}$$

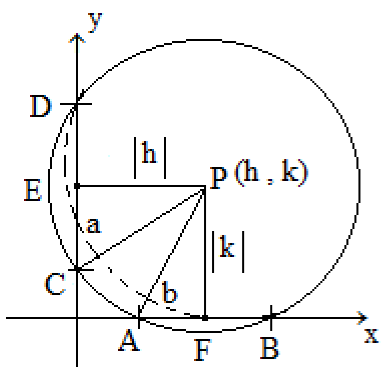
$$\Rightarrow t_1^3 t_2 = -1$$

Q.17

Let two mutually perpendicular lines are x = 0 & y = 0

Let AB = 2a and CD = 2b

Let centre is P(h , k)



$$PE = |h|$$

$$PF = |k|$$

$$PC = \sqrt{h^2 + a^2}$$

$$PA = \sqrt{k^2 + b^2}$$

$$PC^2 = PA^2$$

$$\Rightarrow h^2 + a^2 = k^2 + b^2$$

$$\Rightarrow \boxed{x^2 - y^2 = b^2 - a^2}$$ (rectangular hyperbola)

Q.18

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ and the hyperbola be $xy = a^2$.

Any point on the hyperbola will be $\left(at, \frac{a}{t}\right)$.

Substitute these coordinates in the equation of the circle and rearrange the terms to get

$$a^2t^4 + 2gat^3 + ct^2 + 2fat + a^2 = 0$$

Now $t_1 + t_2 + t_3 + t_4 = -\frac{2g}{a}, \sum t_1t_2t_3 = -\frac{2f}{a}, t_1t_2t_3t_4 = 1$

Hence $\frac{at_1 + at_2 + at_3 + at_4}{4} = -\frac{g}{2}$ & $\frac{\frac{a}{t_1} + \frac{a}{t_2} + \frac{a}{t_3} + \frac{a}{t_4}}{4} - \frac{f}{2}$.

Clearly its midpoint of line joining (0, 0) & (-g, -f)

Q.19

Equation of circle on the line joining foci (ae, 0) and (–ae, 0) as diameter is
$(x - ae)(x + ae) + (y - 0)(y - 0) = 0$
i.e. $x^2 + y^2 = a^2e^2 = a^2 + b^2$ ... (i) [$a^2e^2 = a^2 + b^2$]
Let chord of contact of P (x_1, y_1) touch the circle (i)
Equation of chord of contact of P is [T = 0]
$xx_1/a^2 - yy_1/b^2 = 1$ i.e., $b^2x_1x - a^2y_1y - a^2b^2 = 0$ ... (ii)

$$\therefore \frac{a^2b^2}{\sqrt{(b^4x_1^2 + a^4y_1^2)}} = \pm\sqrt{(a^2 + b^2)}$$

Hence locus of P (x_1, y_1) is $(b^4x^2 + a^4y^2)(a^2 + b^2) = a^4b^4$.

Q.20

Let hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

A normal to it is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$

A is $\left(\frac{(a^2 + b^2)}{a}\sec\theta, 0\right)$

B is $\left(0, \frac{(a^2 + b^2)\tan\theta}{b}\right)$

Let mid – point of AB is P(h , k)

So $h = \frac{(a^2+b^2)\sec\theta}{2a}$; $k = \frac{(a^2+b^2)}{2b}\tan\theta$

Eliminating 'θ' we have

$\Rightarrow (2ah)^2 - (2bk)^2 = (a^2+b^2)^2$

$$\Rightarrow \frac{x^2}{\frac{a^4e^4}{4a^2}} - \frac{y^2}{\frac{a^4e^4}{4b^2}} = 1$$

$$e = \sqrt{1 + \frac{a^4e^4}{4b^2} \times \frac{4a^2}{a^4e^4}}$$

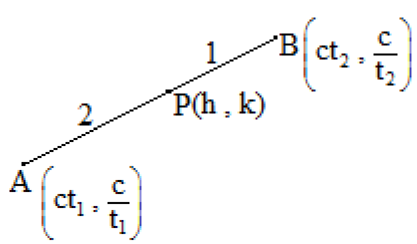
$$e = \sqrt{1 + \frac{1}{e^2-1}}$$

$$e = \frac{e}{\sqrt{e^2-1}}$$

Q.21

Let chord is $A(t_1)$ to $B\ (t_2)$.

Slope AB $= \frac{-1}{t_1+t_2} = 4$



$$h = \frac{(2t_2+t_1)c}{3}$$

$$k = \frac{\left(\frac{2}{t_2} + \frac{1}{t_1}\right)c}{3}$$

We have, $2t_2 + t_1 = \frac{3h}{c}$ & $2t_1 + t_2 = \frac{-3k}{4c}$

Eliminating t_1 & t_2 we get,

$16x^2 + 10xy + y^2 = 2c^2$

Q.22

A tangent to $x^2 = 4ay$ is $x = my + \frac{a}{m}$. It meets $xy = c^2$

So $x = \frac{mc^2}{x} + \frac{a}{m}$

$\Rightarrow mx^2 = m^2c^2 + ax$

$\Rightarrow mx^2 - ax + m^2c^2 = 0$

Let P & Q are (x_1 , y_1) & (x_2 , y_2)

$\Rightarrow$ mid – point be R (h , k)

$h = \frac{x_1 + x_2}{2}$; $k = \frac{y_1 + y_2}{2}$

$x_1 + x_2 = \frac{a}{m}$; $y_1 + y_2 = \frac{-a}{m_2}$

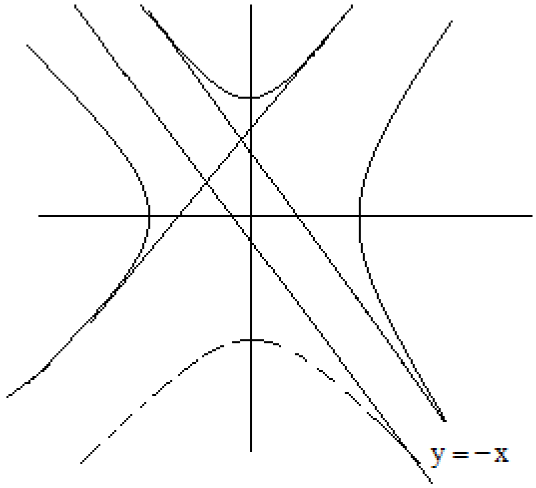
$2h = \frac{a}{m}$; $2k = \frac{-a}{m^2}$

$\frac{4h^2}{a^2} = \frac{-2k}{a}$

$\Rightarrow 2x^2 = -ay \Rightarrow y = \frac{-2x^2}{a}$ (a parabola)

Q.23

The hyperbolas are conjugate to each other so the common tangent will be the ones with slope ± 1



So equation of tangent with slope '1'

$$\boxed{y = x \pm \sqrt{a^2 - b^2}}$$

Let point of tangency is (h , k)

So $x - y - \sqrt{a^2 - b^2} = 0$

$$\frac{xh}{a^2} - \frac{yk}{b^2} - 1 = 0$$

So $\frac{a^2}{h} = \frac{b^2}{k} = + \sqrt{a^2 - b^2}$

Point is $\left(\frac{a^2}{\sqrt{a^2 - b^2}} \;,\; \frac{b^2}{\sqrt{a^2 - b^2}} \right)$

Length is twice of its distance from asymptote y + x = 0

So length is $\frac{\sqrt{2}\left| a^2 + b^2 \right|}{\sqrt{a^2 - b^2}}$

Q.24

Let the normal be $tx - \frac{y}{t} = ct^2 - \frac{c}{t^2}$.

As it passes through $\left(ct_1, \frac{c}{t_1} \right)$ hence $t_1 t - \frac{1}{t_1 t} = t^2 - \frac{1}{t^2}$

$\Rightarrow (t_1 - t)t = \dfrac{t - t_1}{t_1 t^2}$ or $t_1 t^3 = -1$

Q.25

We have $\dfrac{2b^2}{a} = \dfrac{32\sqrt{2}}{5}$

And point of intersection of lines is

$\left.\begin{array}{l} 7x + 13y - 87 = 0 \\ 5x - 8y + 7 = 0 \end{array}\right] \quad \Rightarrow x = 5 \;,\; y = 4$

So we have $\dfrac{25}{a^2} - \dfrac{16}{b^2} = 1$

On solving we have $a = \dfrac{5}{\sqrt{2}}$ & b = 4

Q.26

Hyperbola is $16x^2 - 9y^2 + 32x + 36y - 164 = 0$

$\Rightarrow 16(x+1)^2 - 9(y^2 - 4y + 4) = 144$

$\Rightarrow \dfrac{(x+1)^2}{9} - \dfrac{(y-2)^2}{16} = 1$

Centre $\rightarrow$ (– 1 , 2)

Foci $\rightarrow$ (4 , 2) & (– 6 , 2)

Directrix $\rightarrow$ $x = \dfrac{4}{5}$ & $x = \dfrac{-14}{5}$

Lotus rectum $\rightarrow$ $\dfrac{32}{3}$

Transverse axis $\rightarrow$ 6 ; equation y – 2 = 0

Conjugate axis $\rightarrow$ 8 ; equation x + 1 = 0

Asymptotes $\rightarrow$ 4x – 3y + 10 = 0 & 4x + 3y – 2 = 0

Q.27

Slope of tangent = (– 1)

Equation : $y = - x \pm \sqrt{36 \times 1 - 9}$

$y + x = \pm 3\sqrt{3}$

Q.28

Chord with ‘ θ_1 ’ & ‘ θ_2 ’ as and of extremities.

$$\frac{x}{a} \cos \frac{\theta_1 - \theta_2}{2} - \frac{y}{b} \sin \frac{\theta_1 + \theta_2}{2} = \cos \frac{\theta_1 + \theta_2}{2}$$

It passes through (ae , 0).

So, $e\frac{\cos \theta_1 - \theta_2}{2} = \frac{\cos \theta_1 + \theta_2}{2}$

$$e = \frac{\frac{\cos(\theta_1 + \theta_2)}{2}}{\frac{\cos(\theta_1 - \theta_2)}{2}}$$

$$\frac{e-1}{e+1} = \frac{\cos\left(\frac{\theta_1 + \theta_2}{2}\right) - \cos\left(\frac{\theta_1 - \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right) + \cos\left(\frac{\theta_1 - \theta_2}{2}\right)}$$

$$\frac{e-1}{e+1} = \frac{-2 \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}}{2 \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2}}$$

$$\Rightarrow \boxed{\frac{e-1}{e+1} + \tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2} = 0}$$

Q.29

A line through $\left(0 , \frac{5}{2}\right)$ is $y - \frac{5}{2} = mx$. This is tangent to $3x^2 - 2y^2 = 25$.

So on solving $3x^2 - 2\left(mx + \frac{5}{2}\right)^2 = 25$

$\Rightarrow\ 6x^2 - (3mx+5)^2 = 50$

$\Rightarrow (6-4m^2)x^2 - (20m)x - 75 = 0$

$D = 0 \ \Rightarrow\ 400m^2 + 300(6 - 4m^2) = 0$

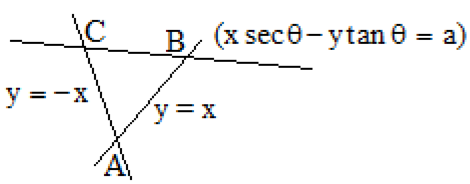
$\Rightarrow\ 800m^2 = 6 \times\ 300$

$\Rightarrow m = \pm\ \frac{3}{2}$

So equation(s) are $\Rightarrow 2y = \pm\ 3x\ +\ 5$

Q.30

A tangent to $x^2 - y^2 = a^2$ is $x\sec\theta - y\tan\theta = a$



A : (0 , 0)

$$B : \left(\frac{a}{\sec\theta - \tan\theta}\ ,\ \frac{a}{\sec\theta - \tan\theta}\right)$$

$$C : \left(\frac{a}{\sec\theta + \tan\theta}\ ,\ \frac{-a}{\sec\theta + \tan\theta}\right)$$

A (0 , 0) : $B(a(\sec\theta + \tan\theta),\ a(\sec\theta + \tan\theta))$

$C(a()\sec\theta - \tan\theta),\ -\ a(\sec\theta\ -\ \tan\theta))$

Area of Δ ABC

$$= \frac{1}{2}\left|[a^2(\sec^2\theta - \tan^2\theta) - a^2(\sec^2\theta - \tan^2\theta)]\right| = a^2$$