ELLIPSE

Exercise – 3

Q.1 $ae = b$ & $a^2 - a^2e^2 = b^2 \Rightarrow a^2 = 2a^2e^2$

$\Rightarrow e = \dfrac{1}{\sqrt{2}}$.

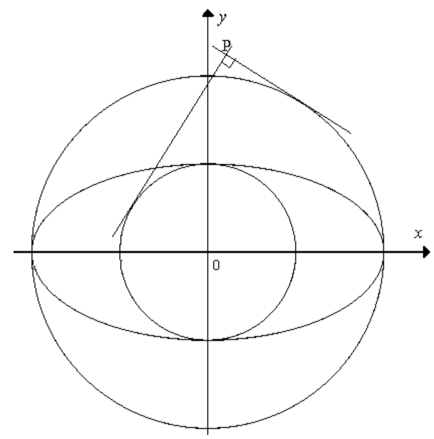
Q.2 Let the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then

the circle on major axis as diameter will be

$x^2 + y^2 = a^2$ and

the circle on minor axis as diameter will be

$x^2 + y^2 = b^2$



Any tangent with slope m to former circle will be

$y = mx + a\sqrt{1+m^2}$ or $y - mx = a\sqrt{1+m^2}$ and

a perpendicular tangent to the later circle will be

$y = -\dfrac{1}{m}x + b\sqrt{1+\dfrac{1}{m^2}}$ or $x + my = b\sqrt{1+m^2}$

From the two equations we get $m = \dfrac{ay - bx}{by + ax}$.

Substituting this value of m in former equation of tangent gives

$x(by+ax) + y(ay-bx) = a\sqrt{(by+ax)^2 + (ay-bx)^2}$

or $x^2 + y^2 = a^2 + b^2$.

Q.3 Vertices of the rectangle will be $P(\theta), Q(\pi-\theta), R(\pi+\theta)$ & $S(-\theta)$.

Hence sides will be of length $2a\cos\theta$ & $2b\sin\theta$.

Perimeter = $2a\cos\theta + 2b\sin\theta \le 2\sqrt{a^2+b^2}$

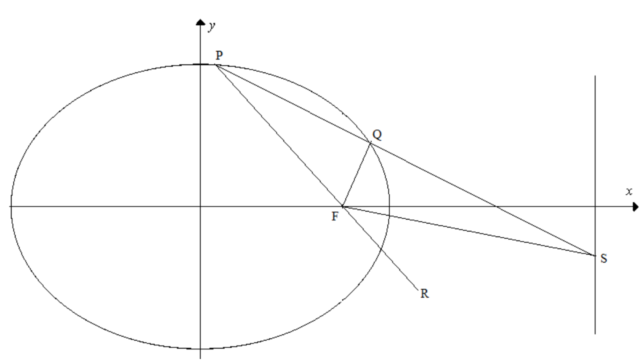
Area = $4ab\cos\theta\sin\theta \le 2ab$.

Q.4 Let eccentric angles of P & Q be α & β, then equation of PQ will be

$\dfrac{x}{a}\cos\dfrac{\alpha+\beta}{2} + \dfrac{y}{b}\sin\dfrac{\alpha+\beta}{2} = \cos\dfrac{\alpha-\beta}{2}$

Hence coordinates of S will be

$\left(\dfrac{a}{e}, \dfrac{b\left(e\cos\dfrac{\alpha-\beta}{2} - \cos\dfrac{\alpha+\beta}{2}\right)}{e\sin\dfrac{\alpha+\beta}{2}}\right)$



Now slope of FS = $\dfrac{b\left(e\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta}{2}\right)}{a\left(1-e^2\right)\sin\frac{\alpha+\beta}{2}}=\dfrac{a\left(e\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta}{2}\right)}{b\sin\frac{\alpha+\beta}{2}}$

Slope of QF = $m_1=\dfrac{b\sin\beta}{a\cos\beta-ae}$ & Slope of PR = $m_2=\dfrac{b\sin\alpha}{a\cos\alpha-ae}$

Let slope of bisector of angle QFR be m, then

$$\frac{m_1-m}{1+m_1m}=\frac{m-m_2}{1+mm_2}\Rightarrow\left(m_1+m_2\right)m^2+2\left(1-m_1m_2\right)m-\left(m_1+m_2\right)=0$$

Now $m_1+m_2=\dfrac{b\sin\alpha}{a\cos\alpha-ae}+\dfrac{b\sin\beta}{a\cos\beta-ae}$

$$=\frac{ab\sin\left(\alpha+\beta\right)-abe\left(\sin\alpha+\sin\beta\right)}{a^2\left(\cos\alpha-e\right)\left(\cos\beta-e\right)}$$

$$=\frac{2ab\sin\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha+\beta}{2}-e\cos\frac{\alpha-\beta}{2}\right)}{a^2\left(\cos\alpha-e\right)\left(\cos\beta-e\right)}$$

and $1-m_1m_2=1-\left(\dfrac{b\sin\alpha}{a\cos\alpha-ae}\right)\left(\dfrac{b\sin\beta}{a\cos\beta-ae}\right)$

$$=\frac{a^2\cos\alpha\cos\beta-a^2e\left(\cos\alpha+\cos\beta\right)+a^2e^2-b^2\sin\alpha\sin\beta}{a^2\left(\cos\alpha-e\right)\left(\cos\beta-e\right)}$$

$$=\frac{\left(a^2-b^2\right)\cos^2\frac{\alpha-\beta}{2}+\left(a^2+b^2\right)\cos^2\frac{\alpha+\beta}{2}-2a^2e\cos\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}+a^2e^2-a^2}{a^2\left(\cos\alpha-e\right)\left(\cos\beta-e\right)}$$

$$=\frac{a^2\left(\cos\frac{\alpha+\beta}{2}-e\cos\frac{\alpha-\beta}{2}\right)^2-b^2\sin^2\frac{\alpha+\beta}{2}}{a^2\left(\cos\alpha-e\right)\left(\cos\beta-e\right)}$$

Let $a\left(\cos\frac{\alpha+\beta}{2}-e\cos\frac{\alpha-\beta}{2}\right)=p$ & $b\sin\frac{\alpha+\beta}{2}=q$

then $m_1+m_2=\dfrac{2pq}{a^2\left(\cos\alpha-e\right)\left(\cos\beta-e\right)}$ & $1-m_1m_2=\dfrac{p^2-q^2}{a^2\left(\cos\alpha-e\right)\left(\cos\beta-e\right)}$

Now $pqm^2+\left(p^2-q^2\right)m-pq=0\Rightarrow m=\dfrac{p}{q}$ & $-\dfrac{q}{p}$

$$\Rightarrow m=\frac{a\left(\cos\frac{\alpha+\beta}{2}-e\cos\frac{\alpha-\beta}{2}\right)}{b\sin\frac{\alpha+\beta}{2}}\ \&\ -\frac{b\sin\frac{\alpha+\beta}{2}}{a\left(\cos\frac{\alpha+\beta}{2}-e\cos\frac{\alpha-\beta}{2}\right)}$$

Hence one of the bisectors of angle QFR is FS.

Q.5 As the ellipse is touching x – axis hence $b^2 = 12$.
(Product of perpendiculars from the foci on any tangent is b^2)
Also $2ae = \sqrt{(-1-5)^2 + (2-6)^2} \Rightarrow a^2e^2 = 13$
Now $a^2 = a^2e^2 + b^2 = 25$
Hence $e = \frac{\sqrt{13}}{5}$.

Q.6 Let radii of the given circles w_1 & w_2 be r_1 & r_2 and that of w be r.
Now AC = $r_1 - r$ & BC = $r_2 + r$, then
AC + BC = $r_1 + r_2$
Hence locus of C is an ellipse foci at A & B and major axis = $r_1 + r_2$.

Q.7 Normal to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at any point P(θ) will be
$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$.
Distance from origin = $\frac{a^2 - b^2}{\sqrt{a^2 \sec^2\theta + b^2 \operatorname{cosec}^2\theta}}$
Now $a^2 \sec^2\theta + b^2 \operatorname{cosec}^2\theta = a^2 \tan^2\theta + b^2 \cot^2\theta + a^2 + b^2$
Further by $A.M. \geq G.M.$
$a^2 \tan^2\theta + b^2 \cot^2\theta \geq 2ab$
Hence $a^2 \sec^2\theta + b^2 \operatorname{cosec}^2\theta \geq a^2 + b^2 + 2ab$
$\Rightarrow \frac{a^2 - b^2}{\sqrt{a^2 \sec^2\theta + b^2 \operatorname{cosec}^2\theta}} \leq a - b$.

Q.8 Tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at any point P(θ) will be
$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$.
It will meet the coordinate axes at $A(a\sec\theta, 0)$ & $B(0, b\operatorname{cosec}\theta)$.
Coordinates of midpoint of AB will be
$x = \frac{a\sec\theta}{2}$ & $y = \frac{b\operatorname{cosec}\theta}{2}$
Eliminating θ gives the required locus as $\frac{a^2}{4x^2} + \frac{b^2}{4y^2} = 1$.

Q.9 Let P be $(a\cos\theta, b\sin\theta)$.
Now F is (ae, 0) hence PF = $a(1 - e\cos\theta)$
Radius of circle on PF as diameter = $\frac{a(1 - e\cos\theta)}{2}$ and center : $\left(\frac{a\cos\theta + ae}{2}, \frac{b\sin\theta}{2}\right)$

Also for auxiliary circle radius = a and center : (0, 0)

Distance between the centers = $\sqrt{\left(\frac{a\cos\theta+ae}{2}\right)^2+\left(\frac{b\sin\theta}{2}\right)^2}$ |

$= \frac{1}{2}\sqrt{a^2\cos^2\theta+2a^2e\cos\theta+a^2e^2+b^2\sin^2\theta}$

$= \frac{1}{2}\sqrt{a^2\cos^2\theta+2a^2e\cos\theta+a^2e^2+b^2-b^2\cos^2\theta}$

$= \frac{1}{2}\sqrt{a^2e^2\cos^2\theta+2a^2e\cos\theta+a^2}$

$= \frac{a(1+e\cos\theta)}{2} = a - \frac{a(1-e\cos\theta)}{2}$

= difference of radii.

Hence circle with PF as diameter touches the auxiliary circle.

Q.10 Let the common tangent be $y = mx + c$.

For being a tangent to the ellipse : $c^2 = a^2m^2 + b^2$

For being a tangent to the circle : $c^2 = r^2(m^2+1)$

Hence $a^2m^2 + b^2 = r^2(m^2+1)$

$\Rightarrow m = \pm\sqrt{\frac{r^2-b^2}{a^2-r^2}} = \tan\theta$

Now parametric coordinates of a point at a distance p from F(ae, 0) on the line RS || PQ will be

$(ae + p\cos\theta, p\sin\theta)$

Substituting these coordinates in the equation of the circle gives

$(ae+p\cos\theta)^2+(p\sin\theta)^2 = r^2$ or $p^2+(2ae\cos\theta)p+a^2e^2-r^2=0$

Roots of this equation will be SF and QF.

As SF & QF are measured in opposite directions from F, hence

RS = difference of roots

$\Rightarrow RS = \frac{\sqrt{4a^2e^2\cos^2\theta-4(a^2e^2-r^2)}}{2}$

$\Rightarrow RS = \sqrt{r^2-a^2e^2\sin^2\theta}$

Now $\tan^2\theta = \frac{r^2-b^2}{a^2-r^2} \Rightarrow \sin^2\theta = \frac{r^2-b^2}{a^2e^2}$

$\Rightarrow RS = b$.

Q.11 Tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at any point P(θ) will be

$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$.

Homogenizing the equation of the auxiliary circle using the equation of tangent gives

$$x^2+y^2=a^2\left(\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}\right)^2 \quad \text{or}$$

$$\left(b^2\sin^2\theta\right)x^2-\left(2ab\sin\theta\cos\theta\right)xy+\left(b^2-a^2\sin^2\theta\right)y^2=0$$

As this pair of straight lines subtends a right angle at the origin hence

coefficient of x^2 + coefficient of $y^2=0$

$$\Rightarrow b^2\sin^2\theta+b^2-a^2\sin^2\theta=0$$

$$\Rightarrow \frac{a^2-b^2}{b^2}=\frac{1}{\sin^2\theta}$$

$$\Rightarrow \frac{a^2-b^2}{a^2}=\frac{1}{1+\sin^2\theta}$$

$$\Rightarrow e=\frac{1}{\sqrt{1+\sin^2\theta}}$$

Q.12 Let P be $\left(a\cos\theta, b\sin\theta\right)$.

Also F_1 & F_2 are $(ae,0)$ & $(-ae,0)$

$F_1F_2=2ae$, $PF_1=a\left(1-e\cos\theta\right)$ & $PF_2=a\left(1+e\cos\theta\right)$

Now $(PF_1-PF_2)^2=4a^2e^2\cos^2\theta$.

Further Tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(\theta)$ will be $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$.

Hence $d=\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$

Now $4a^2\left(1-\frac{b^2}{d^2}\right)=4a^2\left(1-\frac{b^2\cos^2\theta+a^2\sin^2\theta}{a^2}\right)$

$=4\left(a^2-b^2\right)\cos^2\theta=4a^2e^2\cos^2\theta$.

Q.13 Let the extremities of the two semi-diameters be $P(\alpha)$ & $Q(\beta)$, then

$\frac{b\sin\alpha}{a\cos\alpha}\times\frac{b\sin\beta}{a\cos\beta}=-1$ (as the diameters are mutually perpendicular)

$$\Rightarrow b^2\sin\alpha\sin\beta+a^2\cos\alpha\cos\beta=0$$

$$\Rightarrow b^2\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)+a^2\left(\cos(\alpha-\beta)+\cos(\alpha+\beta)\right)=0$$

$$\Rightarrow \left(a^2+b^2\right)\cos^2\frac{\alpha-\beta}{2}+\left(a^2-b^2\right)\cos^2\frac{\alpha+\beta}{2}=a^2 \ldots\text{(i)}$$

Now the chord PQ will be

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$$

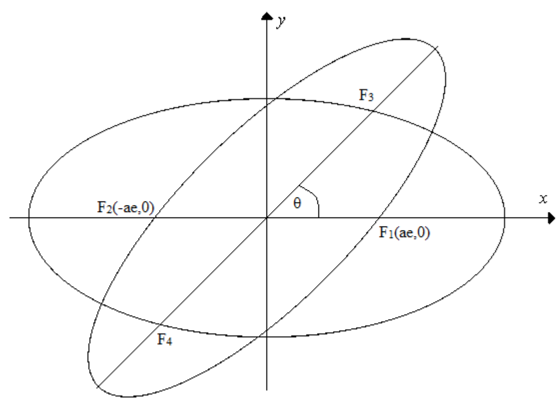
Distance of PQ from the origin is $d = \dfrac{ab\cos\dfrac{\alpha-\beta}{2}}{\sqrt{b^2\cos^2\dfrac{\alpha+\beta}{2}+a^2\sin^2\dfrac{\alpha+\beta}{2}}}$

$$= \frac{ab\cos\dfrac{\alpha-\beta}{2}}{\sqrt{a^2-\left(a^2-b^2\right)\cos^2\dfrac{\alpha+\beta}{2}}}$$

From (i), $d = \dfrac{ab\cos\dfrac{\alpha-\beta}{2}}{\sqrt{\left(a^2+b^2\right)\cos^2\dfrac{\alpha-\beta}{2}}} = \dfrac{ab}{\sqrt{\left(a^2+b^2\right)}}$

Hence PQ touches the circle having radius $\dfrac{ab}{\sqrt{\left(a^2+b^2\right)}}$ and center at the origin.

Q.14 Let length of major axis and eccentricity of one ellipse be 2a & e and that of second ellipse be 2a & e'.



Now $F_1F_3 + F_2F_3 = 2a$

Further if eq. of F_3F_4 is $y = x\tan\theta$, then

F_3 & F_4 will be

$\left(ae'\cos\theta, ae'\sin\theta\right)$ & $\left(-ae'\cos\theta, -ae'\sin\theta\right)$

Now $\sqrt{\left(ae+ae'\cos\theta\right)^2+\left(ae'\sin\theta\right)^2}$

$+\sqrt{\left(ae-ae'\cos\theta\right)^2+\left(ae'\sin\theta\right)^2} = 2a$

$$\Rightarrow \sqrt{a^2e^2+2a^2ee'\cos\theta+a^2e^{2'}} = 2a-\sqrt{a^2e^2-2a^2ee'\cos\theta+a^2e'^2}$$

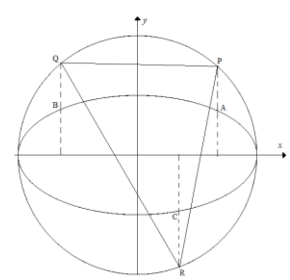
$$\Rightarrow a-aee'\cos\theta = \sqrt{a^2e^2-2a^2ee'\cos\theta+a^2e^{2'}}$$

$$\Rightarrow 1+\left(ee'\cos\theta\right)^2 = e^2+e'^2$$

$$\Rightarrow \cos^2\theta = \frac{e^2+e'^2-1}{e^2e'^2}$$

Q.15 Let the vertices of the triangle be $A(\alpha), B(\beta)$ & $C(\gamma)$.

Consider the triangle formed by corresponding points on auxiliary circle as shown in the adjoining figure.



Now

$$A_{ABC} = \frac{1}{2}\left\|\begin{matrix}1 & a\cos\alpha & b\sin\alpha\\ 1 & a\cos\beta & b\sin\beta\\ 1 & a\cos\gamma & b\sin\gamma\end{matrix}\right\| \;\&\; A_{PQR} = \frac{1}{2}\left\|\begin{matrix}1 & a\cos\alpha & a\sin\alpha\\ 1 & a\cos\beta & a\sin\beta\\ 1 & a\cos\gamma & a\sin\gamma\end{matrix}\right\|$$

Hence $A_{ABC} = \frac{b}{a} A_{PQR}$.

Now Area of triangle PQR will be maximum if its equilateral

i.e. α, β, γ differ by $\frac{2\pi}{3}$.

Now centroid of triangle ABC is

$$\left(\frac{a\cos\alpha + a\cos\left(\alpha + \frac{2\pi}{3}\right) + a\cos\left(\alpha + \frac{4\pi}{3}\right)}{3}, \frac{b\sin\alpha + b\sin\left(\alpha + \frac{2\pi}{3}\right) + b\sin\left(\alpha + \frac{4\pi}{3}\right)}{3} \right) \equiv (0,0)$$

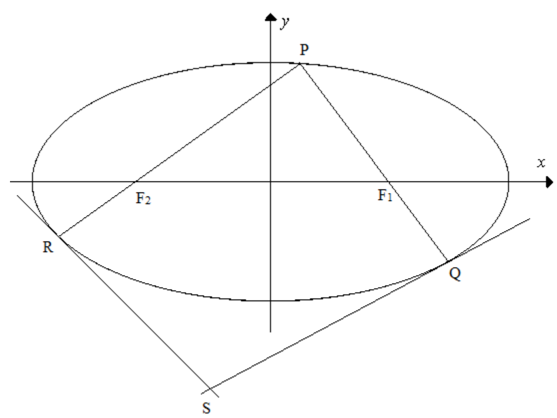
Q.16 Let S be (h, k) and P, Q & R be $(a\cos\theta, b\sin\theta)$, $(a\cos\alpha, b\sin\alpha)$ & $(a\cos\beta, b\sin\beta)$.

Now $h = a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$ & $k = b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$

$$\Rightarrow h = a\frac{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}{1+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}} \ \& \ k = b\frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}} \ldots\text{(i)}$$

Also $\tan\frac{\alpha}{2}\tan\frac{\theta}{2} = \frac{e-1}{e+1}$ & $\tan\frac{\beta}{2}\tan\frac{\theta}{2}$

$$\Rightarrow \tan\frac{\alpha}{2} = \left(\frac{e-1}{e+1}\right)^2 \tan\frac{\beta}{2} \ldots\text{(ii)}$$



From (i) & (ii)

$$\frac{h}{a} = \frac{(e+1)^2 - (e-1)^2\tan^2\frac{\beta}{2}}{(e+1)^2 + (e-1)^2\tan^2\frac{\beta}{2}}$$

$$\Rightarrow \tan^2\frac{\beta}{2} = \left(\frac{e+1}{e-1}\right)^2\left(\frac{a-h}{a+h}\right)$$

$$\& \ \frac{k}{b} = \frac{(e^2+1)2\tan\frac{\beta}{2}}{(e+1)^2 + (e-1)^2\tan^2\frac{\beta}{2}} \Rightarrow \frac{h^2}{a^2} + \frac{(1+e^2)^2}{(1-e^2)^2}\frac{k^2}{b^2} = 1$$

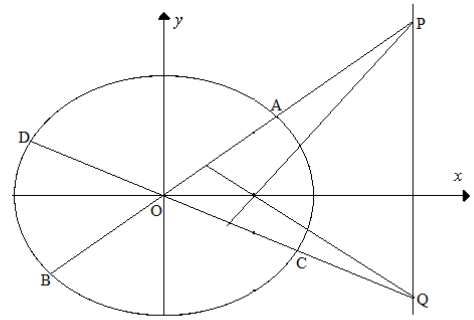
Q.17 Let AB & CD be $y = \frac{mb}{a}x$ & $y = -\frac{b}{ma}x$, then coordinates of P and Q will be

$\left(\frac{a}{e}, \frac{mb}{e}\right)$ & $\left(\frac{a}{e}, -\frac{b}{em}\right)$.

Altitude from P on OQ : $y-\frac{mb}{e}=\frac{ma}{b}\left(x-\frac{a}{e}\right)$

Altitude from O on PQ : y = 0

Hence orthocenter : $\left(\frac{a^2-b^2}{ae},0\right)$ i.e. $(ae,0)$



Q.18 Let three of the sides of quadrilateral be $\frac{x}{a}\cos\frac{\alpha_i+\alpha_{i+1}}{2}+\frac{y}{b}\sin\frac{\alpha_i+\alpha_{i+1}}{2}=\cos\frac{\alpha_i-\alpha_{i+1}}{2}$ for i = 1, 2, 3.

Let direction of three given sides be given by $m_i=-\frac{b}{a}\tan\frac{\alpha_i+\alpha_{i+1}}{2}$ for i = 1, 2, 3.

Clearly direction of fourth side will be uniquely defined as $m_4=-\frac{b}{a}\tan\frac{\alpha_4+\alpha_1}{2}$.

Q.19 Let the equations of AB & CD be $y=m_1x+c_1$ & $y=m_2x+c_2$.

Any curve passing through A, B, C & D will be

$b^2x^2+a^2y^2-a^2b^2+\lambda(m_1x-y+c_1)(m_2x-y+c_2)=0$

If this equation represents a circle, then

$b^2+\lambda m_1m_2=a^2+\lambda$ (coeff. Of x^2 = coeff. Of y^2) …(i)

$m_1+m_2=0$ (coeff. Of xy = 0) …(ii)

Clearly from (ii), AB & CD are equally inclined to coordinate axes.

Q.20 Let $P(\theta),Q\left(\frac{\pi}{2}+\theta\right)$ & $R(\pi+\theta),S\left(\frac{3\pi}{2}+\theta\right)$ represent the end points of diameters PQ & RS.

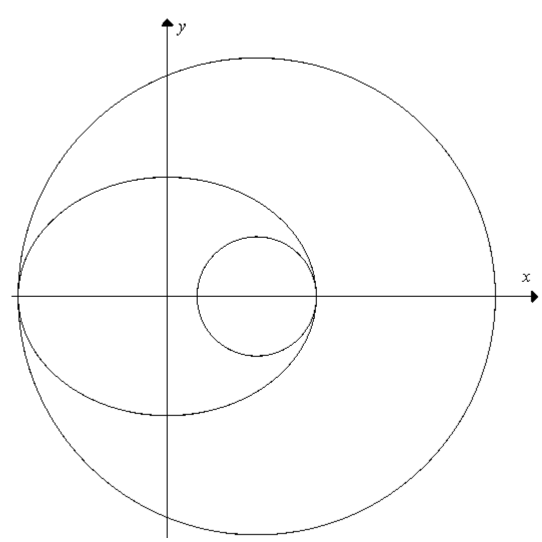
Clearly $PQ^2+RS^2=2(a^2+b^2)$.

Q.21 If any circle having center at (ae, 0) and radius r is touching the ellipse at a point $P(\theta)$, then normal $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$ must pass through (ae, 0) and also distance of $(a\cos\theta,b\sin\theta)$ from (ae, 0) must be r. Hence

(i) $\frac{a^2e}{\cos\theta}-\frac{0}{\sin\theta}=a^2-b^2\Rightarrow\cos\theta=\frac{1}{e}$ (ii)

$a(1-e\cos\theta)=r$

From (i) & (ii) it is clear that no normal except major axis can pass through the focus as



$e < 1$ and hence $1/e > 1$ but $\cos\theta \ngtr 1$.
The required circles must be touching the ellipse at end points of major axis.
Refer the adjoining figure.
$R_1 = a - ae$ & $R_2 = a + ae$.
$R_1 R_2 = a^2 - a^2e^2 = b^2$.

Q.22 Let coordinates of R be (h, k).

equation of chord of contact of $\frac{x^2}{a(a+b)} + \frac{y^2}{b(a+b)} = 1$ w.r.to R will be

$\frac{hx}{a(a+b)} + \frac{ky}{b(a+b)} = 1 \ldots$(i)

Equation of tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at any point will be

$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \ldots$(ii)

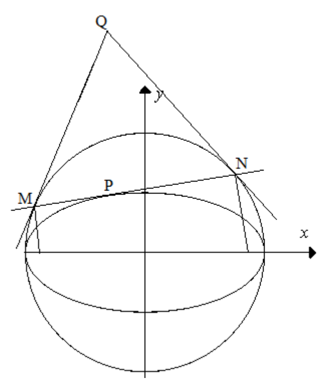
Comparing (i) & (ii) gives $h = (a+b)\cos\theta, k = (a+b)\sin\theta$

Eliminating θ gives $h^2 + k^2 = (a+b)^2$.

Hence R lies on director circle of $\frac{x^2}{a(a+b)} + \frac{y^2}{b(a+b)} = 1$.

$\therefore \angle PRQ = \frac{\pi}{2}$.

Q.23 Feet of perpendiculars from the foci on any tangent lie on the auxiliary circle hence M, N lie on auxiliary circle.
Tangent to the ellipse at P i.e. MN will be chord of contact of Q(h, k) w.r.to the auxiliary circle.



Tangent at P : $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$
Chord of contact of Q : $hx + ky = a^2$.
Comparing the two equations gives

$h = a\cos\theta, k = \frac{a^2}{b}\sin\theta$.

Clearly P & Q have same x coordinate.
Further eliminating θ gives required locus as

$\frac{x^2}{b^2} + \frac{y^2}{a^2} = \frac{a^2}{b^2}$.

Q.24 Let P be $(a\cos\theta, b\sin\theta)$.

Also F_1 & F_2 are $(ae, 0)$ & $(-ae, 0)$

$F_1F_2 = 2ae,\ PF_1 = a(1-e\cos\theta)$ & $PF_2 = a(1+e\cos\theta)$

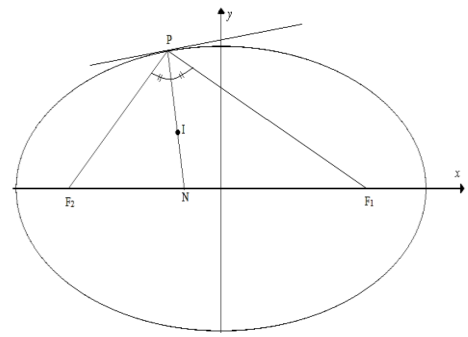
Now in-center will be

$$x = \frac{a(1+e\cos\theta)\times ae + a(1-e\cos\theta)\times(-ae) + 2ae\times a\cos\theta}{2a+2ae}$$

$$\&\ \ y = \frac{a(1+e\cos\theta)\times 0 + a(1-e\cos\theta)\times o + 2ae\times b\sin\theta}{2a+2ae}$$

$$\Rightarrow x = ae\cos\theta,\ y = \frac{be\sin\theta}{1+e} \ \text{ or } \ \cos\theta = \frac{x}{ae}, \sin\theta = \frac{y(1+e)}{be}$$

$$\Rightarrow \frac{x^2}{a^2e^2} + \frac{y^2(1+e)^2}{b^2e^2} = 1.$$



Q.25 Let the given circle be $x^2 + y^2 = a^2$ and A & B be along x – axis.

Also let P be $(a\cos\theta, a\sin\theta)$

Now tangent at A is x = a and tangent at P is $x\cos\theta + y\sin\theta = a$.

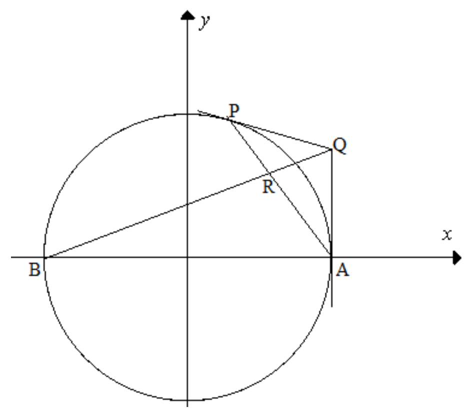
Coordinates of Q will be $\left(a, a\tan\frac{\theta}{2}\right)$

coordinates of B are (-a, 0).

Now equation of AP : $x\cos\frac{\theta}{2} + y\sin\frac{\theta}{2} = a\cos\frac{\theta}{2}$

and equation of BQ : $x\sin\frac{\theta}{2} - 2y\cos\frac{\theta}{2} = -a\sin\frac{\theta}{2}$

Eliminate θ to get required locus as $x^2 + 2y^2 = a^2$.



Q.26 Let PQ & RS be any two mutually perpendicular diameter with eccentric angles of extremities being $P(\alpha), Q(\pi+\alpha), R(\beta)$ & $S(\pi+\beta)$.

Now slope of PQ = $\frac{b}{a}\tan\alpha$ & slope of RS = $\frac{b}{a}\tan\beta$

As $PQ \perp RS \ \therefore\ \tan\alpha\tan\beta = -\frac{a^2}{b^2}$.

Further $PQ = 2\sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}$ & $RS = 2\sqrt{a^2\cos^2\beta + b^2\sin^2\beta}$

Hence $$\frac{1}{PQ^2} + \frac{1}{RS^2} = \frac{1}{4\left(a^2\cos^2\alpha + b^2\sin^2\alpha\right)} + \frac{1}{4\left(a^2\cos^2\beta + b^2\sin^2\beta\right)}$$

$$= \frac{1+\tan^2\alpha}{4\left(a^2 + b^2\tan^2\alpha\right)} + \frac{1+\tan^2\beta}{4\left(a^2 + b^2\tan^2\beta\right)}$$

but $\tan^2\beta = \dfrac{a^4}{b^4\tan^2\alpha}$

$$\Rightarrow \frac{1}{PQ^2}+\frac{1}{RS^2}=\frac{1+\tan^2\alpha}{4\left(a^2+b^2\tan^2\alpha\right)}+\frac{b^4\tan^2\alpha+a^4}{4a^2b^2\left(b^2\tan^2\alpha+a^2\right)}$$

$$=\frac{a^2b^2\left(1+\tan^2\alpha\right)+b^4\tan^2\alpha+a^4}{4a^2b^2\left(b^2\tan^2\alpha+a^2\right)}$$

$$=\frac{\left(a^2+b^2\tan^2\alpha\right)\left(a^2+b^2\right)}{4a^2b^2\left(b^2\tan^2\alpha+a^2\right)}=\frac{a^2+b^2}{4a^2b^2}$$

Q.27 Tangent to $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at $P(\theta)$ will be $\dfrac{x\cos\theta}{a}+\dfrac{y\sin\theta}{b}=1$.

Foot of perpendicular from the origin on this tangent will be given by

$$\frac{x-0}{b\cos\theta}=\frac{y-0}{a\sin\theta}=\frac{ab}{b^2\cos^2\theta+a^2\sin^2\theta}$$

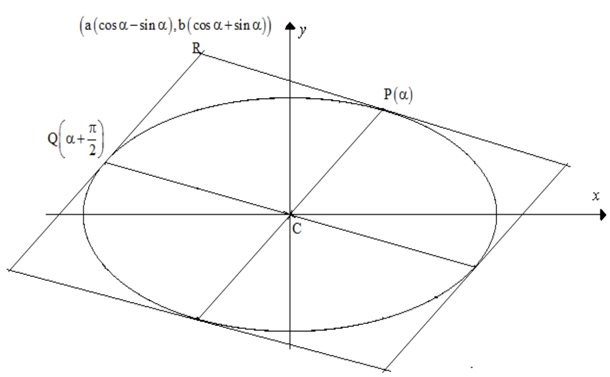
$$\Rightarrow \frac{x}{ab}=\frac{b\cos\theta}{b^2\cos^2\theta+a^2\sin^2\theta}\ \&\ \frac{y}{ab}=\frac{a\sin\theta}{b^2\cos^2\theta+a^2\sin^2\theta}$$

$$\Rightarrow \frac{x^2}{a^2b^2}+\frac{y}{a^2b^2}=\frac{1}{b^2\cos^2\theta+a^2\sin^2\theta}$$

$$\Rightarrow \frac{x}{ab^2\left(\frac{x^2}{a^2b^2}+\frac{y}{a^2b^2}\right)}=\cos\theta\ \&\ \frac{y}{a^2b\left(\frac{x^2}{a^2b^2}+\frac{y}{a^2b^2}\right)}=\sin\theta$$

$$\Rightarrow \frac{x^2}{a^2b^4}+\frac{y^2}{a^4b^2}=\left(\frac{x^2}{a^2b^2}+\frac{y}{a^2b^2}\right)^2.$$

Q.28 Given $P(\alpha)\ \&\ Q\left(\alpha+\dfrac{\pi}{2}\right)$



Now point of intersection of tangents

|at $P(\alpha)\ \&\ Q\left(\alpha+\dfrac{\pi}{2}\right)$:

$R\left(a\left(\cos\alpha-\sin\alpha\right),b\left(\cos\alpha+\sin\alpha\right)\right)$

Required area = $8\times A_{CPR}$

$$=8\times\frac{1}{2}\left\|\begin{matrix}1 & 0 & 0\\ 1 & a\cos\alpha & b\sin\alpha\\ 1 & a\left(\cos\alpha-\sin\alpha\right) & b\left(\cos\alpha+\sin\alpha\right)\end{matrix}\right\|$$

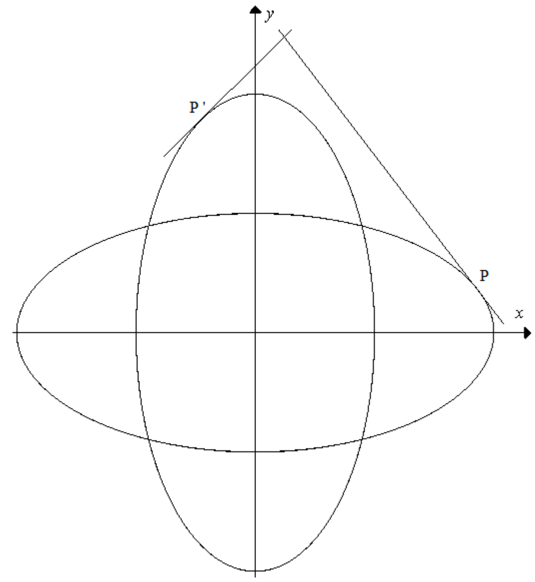
= 4ab

Q.29 Tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $P(\theta)$ will be

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1\ldots(i)$$

After rotation the ellipse will become

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1.$$



Tangent to this ellipse at $P'\left(\frac{\pi}{2}+\theta\right)$ will be

$$-\frac{x\sin\theta}{b}+\frac{y\cos\theta}{a}=1\ldots(ii)$$

From (i) & (ii) we get

$$\cos\theta=\frac{a(x+y)}{x^2+y^2} \text{ \& } \sin\theta=\frac{b(y-x)}{x^2+y^2}$$

Eliminating q gives required locus as

$$a^2(x+y)^2+b^2(x-y)^2=\left(x^2+y^2\right)^2.$$

Q.30 Coordinates of P are $(a\cos\alpha, b\sin\alpha)$ and those of the focus S are (ae , 0)

Hence slope of SP, $\tan\beta=\frac{b\sin\alpha}{a\cos\alpha-ae}$.

$$\Rightarrow \frac{2\tan\frac{\beta}{2}}{1-\tan^2\frac{\beta}{2}}=\frac{b\dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}}{a\dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}-ae}$$

$$\Rightarrow \frac{\tan\frac{\beta}{2}}{1-\tan^2\frac{\beta}{2}}=\frac{\sqrt{\frac{1+e}{1-e}}\tan\frac{\alpha}{2}}{1-\left(\frac{1+e}{1-e}\right)\tan^2\frac{\alpha}{2}}$$

$$\Rightarrow \tan\frac{\beta}{2}=\sqrt{\frac{1+e}{1-e}}\tan\frac{\alpha}{2}.$$

Similar we can prove for the other focus.

Q.31 Let CP & CR be any two mutually perpendicular diameter with eccentric angles of extremities being $P(\alpha) \& R(\beta)$.

Now slope of CP = $\frac{b}{a}\tan\alpha$ & slope of CR = $\frac{b}{a}\tan\beta$

As $CP \perp CR \;\therefore\; \tan\alpha\tan\beta = -\frac{a^2}{b^2}$

$$\Rightarrow b^2\sin\alpha\sin\beta + a^2\cos\alpha\cos\beta = 0.$$

Further $d_1 = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}$ & $d_2 = \sqrt{a^2\cos^2\beta + b^2\sin^2\beta}$

Hence $d_1d_2 = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}\sqrt{a^2\cos^2\beta + b^2\sin^2\beta}$

$$\Rightarrow d_1d_2 = \frac{\sqrt{a^2 + b^2\tan^2\alpha}\sqrt{a^2 + b^2\tan^2\beta}}{\sqrt{1+\tan^2\alpha}\sqrt{1+\tan^2\beta}}$$

$$= \sqrt{a^4\cos^2\alpha\cos^2\beta + b^4\sin^2\alpha\sin^2\beta + a^2b^2\left(\cos^2\alpha\sin^2\beta + \sin^2\alpha\cos^2\beta\right)}$$

$$= \sqrt{\begin{matrix} a^4\cos^2\alpha\cos^2\beta + 2a^2b^2\cos\alpha\cos\beta\sin\alpha\sin\beta + b^4\sin^2\alpha\sin^2\beta \\ +a^2b^2\left(\cos^2\alpha\sin^2\beta + \sin^2\alpha\cos^2\beta\right) - 2a^2b^2\cos\alpha\cos\beta\sin\alpha\sin\beta \end{matrix}}$$

$$= \sqrt{\left(a^2\cos\alpha\cos\beta + b^2\sin\alpha\sin\beta\right)^2 + a^2b^2\left(\cos\alpha\sin\beta - \sin\alpha\cos\beta\right)^2}.$$

Hence $d_1d_2 = ab\left|\sin(\alpha-\beta)\right|$.

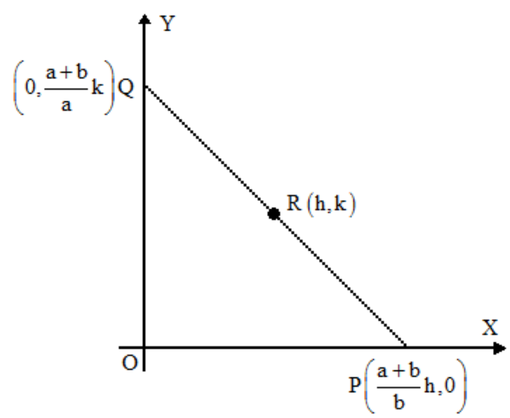
Q.32 From given information let

$\frac{PR}{QR} = \frac{a}{b}$ & $PQ = a + b$.

Hence $\sqrt{\left(\frac{a+b}{b}h\right)^2 + \left(\frac{a+b}{a}k\right)^2} = a + b$

$$\Rightarrow \frac{h^2}{b^2} + \frac{k^2}{a^2} = 1.$$

Required locus is an ellipse.



Q.33 Let $y = mx + \sqrt{a^2m^2 + b^2}$ &

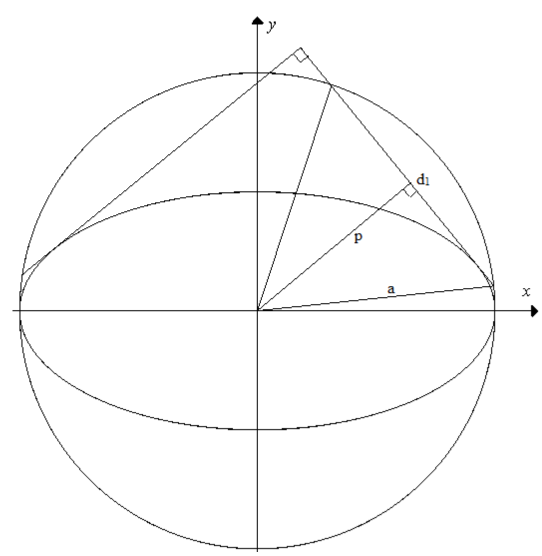
$my = -x + \sqrt{a^2 + b^2m^2}$ be two mutually perpendicular tangents drawn to the ellipse from any point.

Now if the auxiliary circle cuts off a chord of length d_1 on first tangent, then

$d_1^2 = 4\left(a^2 - p^2\right)$, where p is perpendicular distance of the chord from center of the auxiliary circle i.e. (0, 0).

$$\Rightarrow d_1^2 = 4\left(a^2 - \frac{a^2m^2 + b^2}{m^2 + 1}\right) = 4\left(\frac{a^2 - b^2}{m^2 + 1}\right)$$

Similarly for the other tangent



$$\Rightarrow d_2^{\,2} = 4\left(a^2 - \frac{a^2 + b^2m^2}{m^2+1}\right) = 4\left(\frac{\left(a^2-b^2\right)m^2}{m^2+1}\right)$$

Hence $\Rightarrow d_1^{\,2} + d_2^{\,2} = 4\left(\frac{a^2-b^2}{m^2+1}\right) + 4\left(\frac{\left(a^2-b^2\right)m^2}{m^2+1}\right) = (2ae)^2$.

Q.34 Let P be $\left(a\cos\alpha, b\sin\alpha\right)$, then M & N will be $\left(a\cos\alpha, 0\right)$ & $\left(0, b\sin\alpha\right)$.

Equation of MN will be $\frac{x}{a\cos\alpha} + \frac{y}{b\sin\alpha} = 1$

Comparing with $\frac{Ax}{\cos\alpha} + \frac{By}{\sin\alpha} = A^2 - B^2$ (normal to $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$) gives

$\frac{A^2-B^2}{A} = a$ & $\frac{A^2-B^2}{B} = b$

$\Rightarrow A = \frac{ab^2}{b^2-a^2}, B = \frac{a^2b}{b^2-a^2}$.

Hence MN is normal to the ellipse $\frac{x^2}{\left(\frac{ab^2}{b^2-a^2}\right)^2} + \frac{y^2}{\left(\frac{a^2b}{b^2-a^2}\right)^2} = 1$.

Q.35 Let 'd' be the length of referred diameter with one end point at $\left(a\cos\alpha, b\sin\alpha\right)$, then $d^2 = 4\left(a^2\cos^2\alpha + b^2\sin^2\alpha\right)$.

As given $d^2 = \frac{8a^2b^2}{a^2+b^2}$

$\Rightarrow a^2\cos^2\alpha + b^2\sin^2\alpha = \frac{2a^2b^2}{a^2+b^2}$

$\Rightarrow a^4\cos^2\alpha + b^4\sin^2\alpha = a^2b^2\left(\cos^2\alpha + \sin^2\alpha\right)$

$\Rightarrow \left(b^2-a^2\right)b^2\sin^2\alpha = a^2\left(b^2-a^2\right)\cos^2\alpha$

$\Rightarrow \tan\alpha = \pm\frac{a}{b}$

Q.36 Let the common tangent be y = mx + c.

For $\frac{x^2}{9} + \frac{y^2}{4} = 1$: $c^2 = 9m^2 + 4$ …(i)

For $y^2 = 4x$: $c = \frac{1}{m}$ …(ii)

From (i) & (ii), $9m^2 + 4 = \frac{1}{m^2}$ or $9m^4 + 4m^2 - 1 = 0$

$$\Rightarrow m = \pm\sqrt{\frac{\sqrt{13}-2}{9}}$$

Hence common tangents are $\left(\sqrt{13}-2\right)x \pm 3\sqrt{\sqrt{13}-2}\,y = 9$.

Q.37 Let the end points P & Q of conjugate diameters be $P(\theta)$ & $Q\left(\frac{\pi}{2}+\theta\right)$.

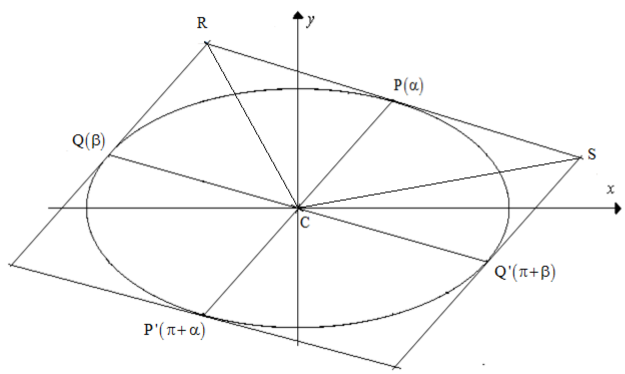
Now point of intersection of tangents at $P\,\&\,Q$:

$$R\left(a\frac{\cos\left(\theta+\frac{\pi}{4}\right)}{\cos\frac{\pi}{4}}, b\frac{\sin\left(\theta+\frac{\pi}{4}\right)}{\cos\frac{\pi}{4}}\right) \text{ i.e. } \frac{x}{a} = \cos\theta - \sin\theta, \frac{y}{b} = \cos\theta + \sin\theta$$

Eliminating θ (square and add) gives required locus as $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 2$.

Q.38 A parallelogram(rectangle) having its vertices lying on an ellipse will be such that the points corresponding to its vertices on auxiliary circle will form a rhombus(square). As diagonals of a rhombus(square) are mutually perpendicular hence diagonals of the referred parallelogram(rectangle) will be conjugate diameters of the ellipse.

Q.39 Given $P(\alpha)\&Q(\beta)$



Now point of intersection of tangents at $P(\alpha)\&Q(\beta)$:

$$R\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$$

Also point of intersection of tangents at $P(\alpha)\&Q(\pi+\beta)$:

$$S\left(-a\frac{\sin\frac{\alpha+\beta}{2}}{\sin\frac{\alpha-\beta}{2}}, b\frac{\cos\frac{\alpha+\beta}{2}}{\sin\frac{\alpha-\beta}{2}}\right)$$

Required area = $4\times\left(A_{CPR}+A_{CPS}\right)$

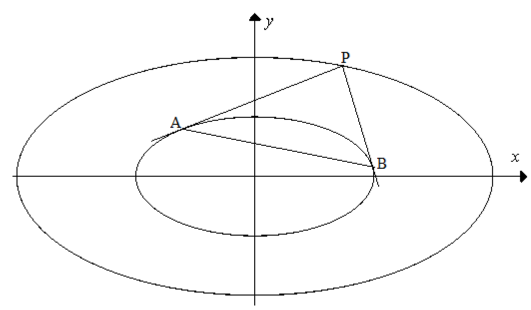
$$= 4\times\frac{1}{2}\left\|\begin{matrix}1 & 0 & 0\\ 1 & a\cos\alpha & b\sin\alpha \\ 1 & a\dfrac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} & b\dfrac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\end{matrix}\right\| + 4\times\frac{1}{2}\left\|\begin{matrix}1 & 0 & 0\\ 1 & a\cos\alpha & b\sin\alpha\\ 1 & -a\dfrac{\sin\frac{\alpha+\beta}{2}}{\sin\frac{\alpha-\beta}{2}} & b\dfrac{\cos\frac{\alpha+\beta}{2}}{\sin\frac{\alpha-\beta}{2}}\end{matrix}\right\|$$

$$= 4ab\left|\frac{\sin\frac{\alpha+\beta}{2}\cos\alpha-\sin\alpha\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right| + 4ab\left|\frac{\cos\frac{\alpha+\beta}{2}\cos\alpha+\sin\alpha\sin\frac{\alpha+\beta}{2}}{\sin\frac{\alpha-\beta}{2}}\right|$$

$$= 4ab\left|\tan\frac{\alpha-\beta}{2}\right| + 4ab\left|\cot\frac{\alpha-\beta}{2}\right|$$

$$= 4ab\left|\operatorname{cosec}(\alpha-\beta)\right|.$$

Q.40 We have to find locus of centroid of triangle formed by tangents of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ drawn from any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=4$ and their respective chord of contact.



Let any point P on the later ellipse be $(2a\cos\theta, 2b\sin\theta)$

Also let A & B be $(a\cos\alpha, b\sin\alpha)$ & $(a\cos\beta, b\sin\beta)$

Now point of intersection of tangents at A & B is $\left(a\dfrac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\dfrac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$, hence

$$2\cos\theta=\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, 2\sin\theta=\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} \Rightarrow \sec^2\frac{\alpha-\beta}{2}=4 \quad \ldots(i)$$

Now centroid of ΔPAB will be

$$G\left(\frac{a(\cos\alpha+\cos\beta+2\cos\theta)}{3}, \frac{b(\sin\alpha+\sin\beta+2\sin\theta)}{3}\right)$$

$$\Rightarrow G\left(\frac{2a\left(\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+\cos\theta\right)}{3},\frac{2b\left(\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+\sin\theta\right)}{3}\right)$$

From (i) we get $G\left(\frac{2a\cos\theta\left(2\cos^2\frac{\alpha-\beta}{2}+1\right)}{3},\frac{2b\sin\theta\left(2\cos^2\frac{\alpha-\beta}{2}+1\right)}{3}\right)$

$\Rightarrow G(a\cos\theta, b\sin\theta)$

Clearly G lies on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Q.41 Point of intersection of tangents at $P(\alpha)\&Q(\beta)$: $R\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

Equation of PQ : $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$

If PQ passes through (ae, 0), then $e\cos\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$

Hence R becomes $\left(\frac{a}{e}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$.

Clearly R lies on the directrics.

Q.42 Normal at any point $(x_1\, y_1)$ to the ellipse $bx^2+a^2y^2=a^2b^2$ is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$

Now normal at $\left(ae, \frac{b^2}{a}\right)$ will be $\frac{a^2x}{ae}-\frac{b^2y}{\frac{b^2}{a}}=a^2-b^2$

$\Rightarrow \frac{a}{e}x - ay = a^2-b^2$

If it passes through (0 , b), then $-ab=a^2-b^2$

$\Rightarrow b^2-a^2=ab$

$\Rightarrow a^2(1-e^2)-a^2=a^2\sqrt{1-e^2} \;\Rightarrow e^4+e^2-1=0$.

Q.43 Let the common tangent be $y = mx + c$.

For being a tangent to the ellipse : $c^2 = a^2m^2 + b^2$

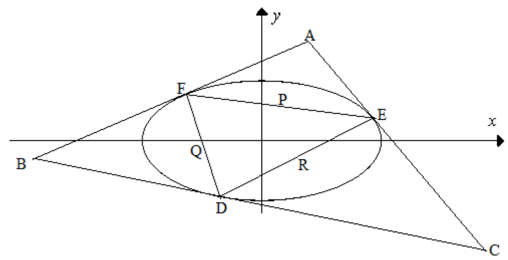
For being a tangent to the circle : $c^2 = r^2(m^2+1)$

Hence $a^2m^2 + b^2 = r^2(m^2+1)$

$$\Rightarrow m = \pm\sqrt{\frac{r^2-b^2}{a^2-r^2}} = \tan\theta$$

Clearly as b < r < a, hence there exists a value of m for every value of r.

Q.44 Let E & F be $(a\cos\alpha, b\sin\alpha)$ & $(a\cos\beta, b\sin\beta)$



Coordinates of A : $\left(a\dfrac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\dfrac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

Also Let P be (h, k)

As P is midpoint of EF hence

EF : $\dfrac{hx}{a^2} + \dfrac{ky}{b^2} = \dfrac{h^2}{a^2} + \dfrac{k^2}{b^2}$ ...(i)

But by equation of chord joining E & F

EF : $\dfrac{x}{a}\cos\dfrac{\alpha+\beta}{2} + \dfrac{y}{b}\sin\dfrac{\alpha+\beta}{2} = \cos\dfrac{\alpha-\beta}{2}$ ...(ii)

Comparing (i) & (ii) gives $\dfrac{h}{a\cos\frac{\alpha+\beta}{2}} = \dfrac{k}{b\sin\frac{\alpha+\beta}{2}} = \dfrac{\frac{h^2}{a^2}+\frac{k^2}{b^2}}{\cos\frac{\alpha-\beta}{2}}$

$$\Rightarrow \frac{h}{\frac{h^2}{a^2}+\frac{k^2}{b^2}} = \frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \frac{k}{\frac{h^2}{a^2}+\frac{k^2}{b^2}} = \frac{b\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$$

$$\Rightarrow \cos^2\frac{\alpha-\beta}{2} = \frac{b^2h^2+a^2k^2}{a^2b^2} \text{ \& } \tan\frac{\alpha+\beta}{2} = \frac{ak}{bh} \text{ or } \cos^2\frac{\alpha+\beta}{2} = \frac{b^2h^2}{b^2h^2+a^2k^2}$$

$$\Rightarrow h = a\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}, k = b\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

Now eq. of AP : $ay\cos\dfrac{\alpha+\beta}{2} = bx\sin\dfrac{\alpha+\beta}{2}$

Clearly AP, BQ & CR will be concurrent at the origin.

Q.45 Given parabola is $y^2 = (4a\cos\alpha)x$.

Let the feet of normals be $P(t_1), Q(t_2)$ & $R(t_3)$.

As the normals are concurrent at a point (h, $b\sin\alpha$) lying on $y = b\sin\alpha$, Hence $t_1+t_2+t_3=0,\ t_1t_2+t_2t_3+t_3t_1=\dfrac{2a\cos\alpha-h}{a\cos\alpha}$ & $t_1t_2t_3=\dfrac{b\sin\alpha}{a\cos\alpha}$ ...(i)

Now point of intersection of tangents at P & Q : $A\left(a\cos\alpha\, t_1t_2, a\cos\alpha\left(t_1+t_2\right)\right)$

and point of intersection of tangents at Q & R : $B\left(a\cos\alpha\, t_2t_3, a\cos\alpha\left(t_2+t_3\right)\right)$

Slopes of respective altitudes : $-t_3$ & $-t_1$

Hence altitude from A : $y-a\cos\alpha\left(t_1+t_2\right)=-t_3\left(x-a\cos\alpha\, t_1t_2\right)$

and altitude from B : $y-a\cos\alpha\left(t_2+t_3\right)=-t_1\left(x-a\cos\alpha\, t_2t_3\right)$

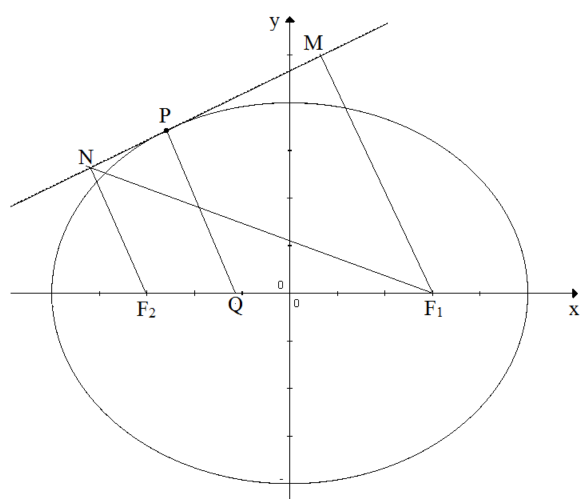
Solving together gives the orthocenter as

$x=-a\cos\alpha$ & $y=\left(t_1+t_2+t_3+t_1t_2t_3\right)a\cos\alpha$

From (i), $x=-a\cos\alpha$ & $y=b\sin\alpha$.

Eliminating α gives required locus as $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Q.46 Let P be $\left(a\cos\alpha, b\sin\alpha\right)$



Now tangent at P $\dfrac{x\cos\alpha}{a}+\dfrac{y\sin\alpha}{b}=1$ and

normal at P $\dfrac{ax}{\cos\alpha}-\dfrac{by}{\sin\alpha}=a^2-b^2$

Coordinates of Q : $\left(\dfrac{a^2-b^2}{a}\cos\alpha,\ 0\right)$

Midpoint of PQ : $R\left(\dfrac{a\left(e^2+1\right)\cos\alpha}{2},\dfrac{b\sin\alpha}{2}\right)$

Further foot of perpendicular on tangent at P from (–ae, 0) will be given by

$$\frac{x+ae}{b\cos\alpha}=\frac{y}{a\sin\alpha}=\frac{ab\left(e\cos\alpha+1\right)}{b^2\cos^2\alpha+a^2\sin^2\alpha}$$

But $b^2\cos^2\alpha+a^2\sin^2\alpha=a^2\left(1-e^2\cos^2\alpha\right)$, hence

$$\frac{x+ae}{b\cos\alpha}=\frac{y}{a\sin\alpha}=\frac{b}{a\left(1-e\cos\alpha\right)}$$

$$\Rightarrow N\left(\frac{a\left(\cos\alpha-e\right)}{\left(1-e\cos\alpha\right)},\frac{b\sin\alpha}{\left(1-e\cos\alpha\right)}\right)$$

Now prove that F_1, N & R are collinear.

Q.47 Any tangent to $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with slope m : $y=mx+\sqrt{a^2m^2+b^2}$

Any tangent to $\dfrac{x^2}{a^2+k}+\dfrac{y^2}{b^2+k}=1$ with slope $-\dfrac{1}{m}$: $my+x=\sqrt{a^2+k+\left(b^2+k\right)m^2}$

From the two equations we get

$\left(x^2-a^2\right)m^2-2xym+y^2-b^2=0$ and

$\left(y^2-b^2-k\right)m^2+2xym+x^2-a^2-k=0$

Comparing the two quadratic equations for common values of m gives

$x^2-a^2=-y^2+b^2+k$ or $x^2+y^2=a^2+b^2+k$.

Q.48 Let P & Q be $(a\cos\alpha, b\sin\alpha)$ & $(-a\sin\alpha, b\cos\alpha)$

Circles on OP & OQ as diameters will be

$x(x-a\cos\alpha)+y(y-b\sin\alpha)=0$ & $x(x+a\sin\alpha)+y(y-b\cos\alpha)=0$

or $x^2+y^2=ax\cos\alpha+by\sin\alpha$ & $x^2+y^2=by\cos\alpha-ax\sin\alpha$

Square and add to eliminate α and get the required locus as

$2\left(x^2+y^2\right)^2=a^2x^2+b^2y^2$.

Q.49 Given that center is at (1, 2) and focus is at (6, 2), hence major axis is along y = 2 and minor axis along x = 1.

Also ae = 5.

Equation of ellipse : $\frac{(x-1)^2}{a^2}+\frac{(y-2)^2}{b^2}=1$

As it passes through (4, 6) hence

$\frac{(4-1)^2}{a^2}+\frac{(6-2)^2}{b^2}=1 \Rightarrow \frac{9}{a^2}+\frac{16}{a^2\left(1-e^2\right)}=1$

$\Rightarrow \frac{9}{a^2}+\frac{16}{a^2-25}=1$

$\Rightarrow a^4-50a^2+225=0 \Rightarrow a^2=45$ & $b^2=20$.

Q. 50 Let P be $(a\cos\theta, a\sin\theta)$.

Tangent to the given circle at P : $x\cos\theta+y\sin\theta=a$

Tangent at A(a, 0) : x = a

Point of intersection of the two tangents : $T\left(a, \frac{a(1-\cos\theta)}{\sin\theta}\right)$

Now B is (–a, 0), hence

equation of BT : $y=\frac{1-\cos\theta}{2\sin\theta}(x+a)$ i.e. $x-2y\cot\frac{\theta}{2}+a=0 \ldots$(i)

Equation of AP : $y=\frac{\sin\theta}{\cos\theta-1}(x-a)$ i.e. $x+y\tan\frac{\theta}{2}-a=0 \ldots$(ii)

From (i) & (ii), eliminating $\tan\frac{\theta}{2}$ gives $\frac{x^2}{2a^2}+\frac{y^2}{a^2}=1$.

Now $e=\sqrt{\frac{2a^2-a^2}{2a^2}}=\frac{1}{\sqrt{2}}$.

Q.51 Tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at any point P(α) will be

$$\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=1.$$

Homogenizing the equation of the auxiliary circle using the equation of tangent gives

$$x^2+y^2=a^2\left(\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}\right)^2 \quad \text{or}$$

$$\left(b^2\sin^2\alpha\right)x^2-\left(2ab\sin\alpha\cos\alpha\right)xy+\left(b^2-a^2\sin^2\alpha\right)y^2=0$$

As this pair of straight lines subtends a right angle at the origin hence

coefficient of x^2 + coefficient of $y^2=0$

$$\Rightarrow b^2\sin^2\alpha+b^2-a^2\sin^2\alpha=0$$

$$\Rightarrow \frac{a^2-b^2}{b^2}=\frac{1}{\sin^2\alpha}$$

$$\Rightarrow \frac{a^2-b^2}{a^2}=\frac{1}{1+\sin^2\alpha}$$

$$\Rightarrow e=\frac{1}{\sqrt{1+\sin^2\alpha}}$$

Q.52 Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

As it passes through (– 3, 1) & (2, –2) hence

$$\frac{9}{a^2}+\frac{1}{b^2}=1 \ \& \ \frac{4}{a^2}+\frac{4}{b^2}=1$$

$$\Rightarrow a^2=\frac{32}{3} \ \& \ b^2=\frac{32}{5}.$$

Required ellipse is $3x^2+5y^2=32$.

Q.53 Let the two points on major axis be P(c, 0) & (-c, 0)

Further let equation of chord passing through (c, 0) be

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)|$$

$$\Rightarrow \frac{c}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+0=\cos\left(\frac{\alpha-\beta}{2}\right) \ \text{Or} \ \frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}=\frac{a}{c}$$

Taking componendo and dividendo

$$\frac{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}{-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}=\frac{a+c}{a-c} \quad \text{Or } \tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{c-a}{c+a}$$

Similarly for the chord passing through (-c, 0) we will get $\tan\frac{\gamma}{2}\tan\frac{\delta}{2}=\frac{c+a}{c-a}$

Hence $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2}=1$.

Q.54 Equation of normal at P(α) : $2ax\sin\alpha-2by\cos\alpha=\left(a^2-b^2\right)\sin 2\alpha$

Equation of normal at Q(β) : $2ax\sin\beta-2by\cos\beta=\left(a^2-b^2\right)\sin 2\beta$

Equation of normal at R(γ) : $2ax\sin\gamma-2by\cos\gamma=\left(a^2-b^2\right)\sin 2\gamma$

As the normals are concurrent, hence

$$\begin{vmatrix} 2a\sin\alpha & -2b\cos\alpha & \left(a^2-b^2\right)\sin 2\alpha \\ 2a\sin\beta & -2b\cos\beta & \left(a^2-b^2\right)\sin 2\beta \\ 2a\sin\gamma & -2b\cos\gamma & \left(a^2-b^2\right)\sin 2\gamma \end{vmatrix}=0 \text{ or}$$

$$-4ab\left(a^2-b^2\right)\begin{vmatrix} \sin\alpha & \cos\alpha & \sin 2\alpha \\ \sin\beta & \cos\beta & \sin 2\beta \\ \sin\gamma & \cos\gamma & \sin 2\gamma \end{vmatrix}=0.$$

Q.55 The circle touching the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the points $\left(ae,\pm\frac{2b^2}{a}\right)$ will have its center on x – axis.

Now normals to the ellipse at $\left(ae,\pm\frac{2b^2}{a}\right)$ will be

$$\frac{x-ae}{ae}a^2=\frac{y-\frac{2b^2}{a}}{\frac{2b^2}{a}}b^2 \;\;\&\;\; \frac{x-ae}{ae}a^2=\frac{y+\frac{2b^2}{a}}{-\frac{2b^2}{a}}b^2$$

Solving these gives center of the circle as $\left(ae-\frac{b^2e}{a},0\right)$ or $\left(ae^3,0\right)$

Further radius = $\sqrt{\left(ae^3-ae\right)^2+\frac{b^4}{a^2}}=a\left(1-e^2\right)\sqrt{e^2+1}$.

Q.56 Let the required tangent be $\frac{x\cos\theta}{4}+\frac{y\sin\theta}{3}=1$

Now given $\frac{4}{\cos\theta}=\frac{3}{\sin\theta}$

Hence $\cos\theta = \pm\frac{4}{5}$ & $\sin\theta = \pm\frac{3}{5}$

Required lines are $x + y = \pm 5$.

Q.57 Let the point of intersection of tangents be (h, k), then the corresponding chord of contact will be $\frac{hx}{a^2} + \frac{ky}{b^2} = 1$.

Homogenizing equation of ellipse using this equation of chord gives

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{hx}{a^2} + \frac{ky}{b^2}\right)^2$$

As this pair of lines subtends a right angle at the origin hence

Coefficient of x^2 + coefficient of y^2 = 0

$$\Rightarrow \frac{h^2 - a^2}{a^4} + \frac{k^2 - b^2}{b^4} = 0$$

Hence the required locus is $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}$.

Q.58 Tangents to $\frac{x^2}{18} + \frac{y^2}{32} = 1$ with slope $-\frac{4}{3}$ will be

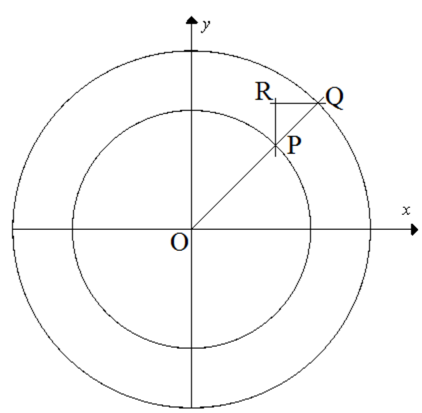
$y = -\frac{4}{3}x \pm \sqrt{18 \times \frac{16}{9} + 32}$ i.e. $4x + 3y = \pm 8$.

Now legs of the triangle OAB will be 2 & $\frac{8}{3}$

hence area = $\frac{1}{2} \times 2 \times \frac{8}{3} = \frac{8}{3}$.

Q.59 Let P be $(a\cos\alpha, a\sin\alpha)$ & Q be $(b\cos\alpha, b\sin\alpha)$,

then R will be $(a\cos\alpha, b\sin\alpha)$



Eliminating a between x & y coordinates of R gives required locus as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Clearly locus of R is an ellipse touching the two circles {Touching inner circle at $(0, \pm a)$ and outer circle at $(\pm b, 0)$}

Also If the foci of this ellipse lie on the inner circle, then a = be.

But $e^2 = 1 - \frac{a^2}{b^2} \Rightarrow e = \frac{a}{b} = \frac{1}{\sqrt{2}}$.

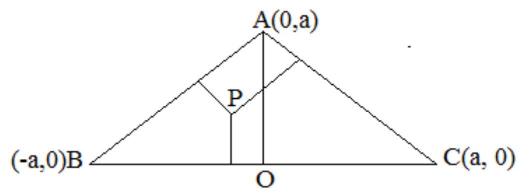
Q.60 Let P be (h, k)

Distance of P from BC = |k|

Equations of AB & AC are

x – y = – a & x + y = a.

Distances of P from AB & AC



$\frac{|h-k+a|}{\sqrt{2}}$ & $\frac{|h+k-a|}{\sqrt{2}}$.

Further for any point inside the triangle x + y < a & x – y + a > 0.

As given $k^2 = \frac{(k-a)^2 - h^2}{4}$

Hence required locus is $x^2 + 3y^2 + 2ay - a^2 = 0$ or $\frac{x^2}{3} + \left(y + \frac{a}{3}\right)^2 = \frac{10a^2}{27}$

Clearly the locus is an ellipse passing through B & C and e = $\sqrt{\frac{2}{3}}$.

Q.61 Let midpoint of the chord be (h, k), them by T = S_1 equation of the chord will be

$\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$.

As this chord is drawn from (a, -b) and (h, k) lies on x + y = b, hence

$\frac{ah}{a^2} + \frac{-b(b-h)}{b^2} = \frac{h^2}{a^2} + \frac{(b-h)^2}{b^2}$

$\Rightarrow (a^2 + b^2)h^2 - (3a^2b + ab^2)h + 2a^2b^2 = 0$

For two distinct values of h, discriminant > 0

$\Rightarrow (3a + b)^2 > 8(a^2 + b^2)$

$\Rightarrow (a - 7b)(a + b) > 0$

$\Rightarrow a > 7b$

Q.62 Let y = mx + c be the common tangent.

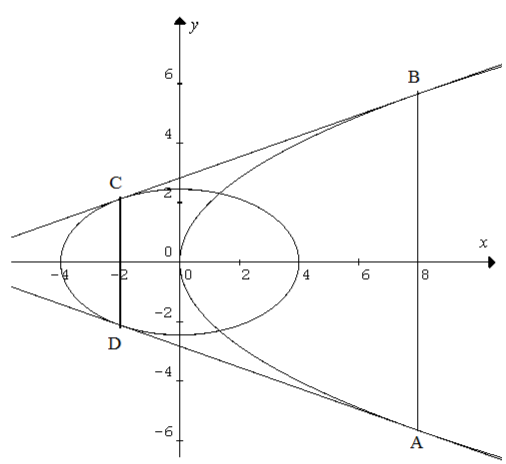
For $\frac{x^2}{16} + \frac{y^2}{6} = 1$, $c^2 = 16m^2 + 6 \ldots$(i)

and for $y^2 = 4x$, $c = \frac{1}{m} \ldots$(ii)

From (i) & (ii) we get

$\frac{1}{m^2} = 16m^2 + 6$ or $16m^4 + 6m^2 - 1 = 0$

Hence $m = \pm\frac{1}{2\sqrt{2}}$.



Therefor the common tangents are

$x \pm 2\sqrt{2}y + 8 = 0$.

Now tangent to $y^2 = 4x$ at a point (h, k) will be $2x - ky + 2h = 0$.

Comparing this with the equations of common tangents gives A & B as $\left(8, \pm 2\sqrt{2}\right)$.

Similarly tangent to $\frac{x^2}{16} + \frac{y^2}{6} = 1$ will be $3hx + 8ky = 48$.

Comparing this with the equations of common tangents gives C & D as $\left(-2, \pm\frac{3\sqrt{2}}{2}\right)$

Hence the quadrilateral ABCD is a trapezium as shown.

AB = $4\sqrt{2}$, CD = $3\sqrt{2}$ & distance between AB & CD = 10.

Required area = $\frac{1}{2} \times \left(4\sqrt{2} + 3\sqrt{2}\right) \times 10 = 35\sqrt{2}$.

Q.63 Equation of normal at P(θ) : $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2e^2$

Coordinates of G : $\left(ae^2\cos\theta, 0\right)$

Coordinates of g : $\left(0, -\frac{a^2e^2}{b}\sin\theta\right)$

Hence $CG^2 = a^2e^4\cos^2\theta$ & $Cg^2 = \frac{a^4e^4}{b^2}\sin^2\theta$.

Now $a^2\left(CG^2\right) + b^2\left(Cg^2\right) = a^4e^4\cos^2\theta + a^4e^4\sin^2\theta$

$\Rightarrow a^2\left(CG^2\right) + b^2\left(Cg^2\right) = \left(a^2 - b^2\right)^2$.

Further $CG = ae^2\cos\theta = e^2\left(a\cos\theta\right) = e^2\left(CN\right)$.

Q.64 Any point on a line of slope tan θ, passing through S(ae, 0) at a distance r from S will be $\left(ae + r\cos\theta, r\sin\theta\right)$.

These coordinates will satisfy equation of the ellipse for two value of |r|
i.e. $|r_1|$ = PA & $|r_2|$ = PB,
where A & B are points of intersection of this chord with the ellipse.
Substituting these coordinates in the equation of ellipse gives

$b^2\left(ae + r\cos\theta\right)^2 + a^2\left(r\sin\theta\right)^2 = a^2b^2$ or

$\left(a^2\sin^2\theta + b^2\cos^2\theta\right)r^2 + \left(2ab^2e\cos\theta\right)r + a^2b^2\left(e^2 - 1\right) = 0$

Now length of chord will be $|r_1| + |r_2|$ i.e. $|r_1 - r_2|$.

Hence length = $\sqrt{\left(r_1 + r_2\right)^2 - 4r_1r_2}$

$$= \frac{2ab\sqrt{b^2e^2\cos^2\theta-(e^2-1)(a^2\sin^2\theta+b^2\cos^2\theta)}}{a^2\cos^2\theta+b^2\sin^2\theta}$$

$$= \frac{2ab\sqrt{b^2e^2\cos^2\theta-a^2(e^2-1)(1-e^2\cos^2\theta)}}{a^2\cos^2\theta+b^2\sin^2\theta}=\frac{2ab^2}{a^2\cos^2\theta+b^2\sin^2\theta}.$$

Q.65 Let P be $(a\cos\theta, b\sin\theta)$, then equation of tangent will be

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

Coordinates of T : $(a\sec\theta, 0)$

Coordinates of N : $(a\cos\theta, 0)$

Circle on TN as diameter : $(x-a\cos\theta)(x-a\sec\theta)+y^2=0$

or $x^2+y^2-a(\cos\theta+\sec\theta)x+a^2=0$

Now $g_1g_2+f_1f_2=a(\cos\theta+\sec\theta)\times 0+0\times 0=0$ & $\frac{c_1+c_2}{2}=\frac{a^2-a^2}{2}=0$

Clearly the two circles are orthogonal.

Q.66 Let the tangents from $T(x_1, y_1)$ be TP & TQ and normals at these points be NP & NQ, N being (h, k).

But (x_1, y_1) lies on the director circle.

Also the tangents and normals will form a cyclic quadrilateral hence (h, k) will also lie on the director circle.

Further the director circle will be drawn on TN as diameter as $\angle TPN=\angle TQN=\frac{\pi}{2}$.

Hence $\frac{h+x_1}{2}=\frac{k+y_1}{2}=0$ i.e. $\frac{h}{x_1}=\frac{k}{y_1}$.

Q.67 Let the moving point be (h, k).

The eq. of chord of contact will be $\frac{hx}{a^2}+\frac{ky}{b^2}=1$

As it is touching $x^2+y^2=c^2$, hence its distance from (0, 0) must be c.

$$\Rightarrow \frac{1}{\sqrt{\frac{h^2}{a^4}+\frac{k^2}{b^4}}}=c \text{ or } \frac{h^2}{a^4}+\frac{k^2}{b^4}=\frac{1}{c^2}$$

Hence the required locus is $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}$.

Q.68 Let y = mx + c be the common tangent, then

for $\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=1$, $c^2=a_1{}^2m^2+b_1{}^2$ or $c^2-a_1{}^2m^2=b_1{}^2$ …(i) for $\dfrac{x^2}{a_2^2}+\dfrac{y^2}{b_2^2}=1,\ c^2=a_2{}^2m^2+b_2{}^2$ or $c^2-a_2{}^2m^2=b_2{}^2$ …(ii) and

for $\dfrac{x^2}{a_3^2}+\dfrac{y^2}{b_3^2}=1,\ c^2=a_3{}^2m^2+b_3{}^2$ or $c^2-a_3{}^2m^2=b_3{}^2$ …(iii)

Now for (i), (ii) & (iii) to have a simultaneous solution in (c, m)

$$\begin{vmatrix} a_1^2 & b_1^2 & 1 \\ a_2^2 & b_2^2 & 1 \\ a_3^2 & b_3^2 & 1 \end{vmatrix}=0.$$

Q.69 Let coordinates of P be $(a\cos\theta, b\sin\theta)$

Now equation of normal at P : $\dfrac{ax}{\cos\theta}-\dfrac{by}{\sin\theta}=a^2-b^2$

The normal meets x – axis at G, hence

Coordinates of G : $\left(\dfrac{a^2-b^2}{a}\cos\theta, 0\right)$

Also let coordinates of Q be (h, k), then P is midpoint of GQ

$$\Rightarrow a\cos\theta=\frac{\dfrac{a^2-b^2}{a}\cos\theta+h}{2}, b\sin\theta=\frac{0+k}{2}$$

$$\Rightarrow h=\left(\frac{a^2+b^2}{a}\right)\cos\theta, k=2b\sin\theta$$

Eliminating θ gives $\Rightarrow \dfrac{h^2}{\left(\dfrac{a^2+b^2}{a}\right)^2}+\dfrac{k^2}{4b^2}=1.$

Hence the required locus is the ellipse $\dfrac{x^2}{\left(\dfrac{a^2+b^2}{a}\right)^2}+\dfrac{y^2}{4b^2}=1.$

Clearly eccentricity is $\sqrt{\dfrac{\left(\dfrac{a^2+b^2}{a}\right)^2-4b^2}{\left(\dfrac{a^2+b^2}{a}\right)^2}}$ or $\dfrac{a^2-b^2}{a^2+b^2}$.

Now tangent to given ellipse at P : $\dfrac{x\cos\theta}{a}+\dfrac{y\sin\theta}{b}=1$

and tangent to the locus at Q : $\dfrac{x\cos\theta}{\dfrac{a^2+b^2}{a}}+\dfrac{y\sin\theta}{2b}=1$

$$\Rightarrow \sin\theta = \frac{2b^3}{(b^2-a^2)y}, \ \cos\theta = \frac{a(a^2+b^2)}{(a^2-b^2)x}$$

Eliminating q gives the locus as $\frac{4b^6}{y^2} + \frac{a^2(a^2+b^2)^2}{x^2} = (a^2-b^2)^2$.

Q.70 Let P & Q be $(a\cos\theta, b\sin\theta)$ & $(a\cos\theta, a\sin\theta)$

Tangent to the ellipse at P : $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

Coordinates of T : $(a\sec\theta, 0)$

Equation of QT : $\frac{y}{x - a\sec\theta} = \frac{a\sin\theta}{a\cos\theta - a\sec\theta}$ or $x\cos\theta + y\sin\theta = a$

Clearly QT is tangent to the auxiliary circle.

Q.71 Let coordinates of P be $(a\cos\theta, b\sin\theta)$

Slope of PA : $\frac{b\sin\theta}{a(\cos\theta-1)}$

Slope of PA' : $\frac{b\sin\theta}{a(\cos\theta+1)}$

Equation of line perpendicular to PA, passing through P :

$$y - b\sin\theta = -\frac{a(\cos\theta-1)}{b\sin\theta}(x - a\cos\theta)$$

Point where it meets x – axis : $Q\left(\frac{b^2\sin^2\theta + a^2\cos\theta(\cos\theta-1)}{a(\cos\theta-1)}, 0\right)$

Equation of line perpendicular to PA', passing through P :

$$y - b\sin\theta = -\frac{a(\cos\theta+1)}{b\sin\theta}(x - a\cos\theta)$$

Point where it meets x – axis : $R\left(\frac{b^2\sin^2\theta + a^2\cos\theta(\cos\theta+1)}{a(\cos\theta+1)}, 0\right)$

Now $\ell(QR) = \frac{b^2\sin^2\theta + a^2\cos\theta(\cos\theta+1)}{a(\cos\theta+1)} - \frac{b^2\sin^2\theta + a^2\cos\theta(\cos\theta-1)}{a(\cos\theta-1)}$

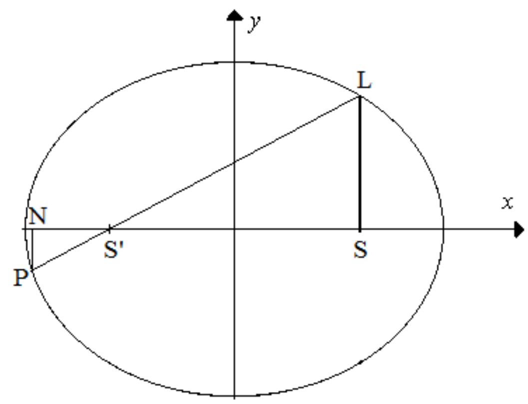
$$\Rightarrow \ell(QR) = \frac{b^2\sin^2\theta((\cos\theta-1)-(\cos\theta+1))}{a(\cos^2\theta-1)} \Rightarrow \ell(QR) = \frac{2b^2}{a}.$$

Q.72 Let eccentric angles of L & P be θ & α.

Now LP is a focal chord, hence $\tan\frac{\theta}{2}\tan\frac{\alpha}{2} = \frac{1+e}{1-e}$.

Comparing $\left(ae, \frac{b^2}{a}\right)$ with $(a\cos\theta, b\sin\theta)$ gives $\cos\theta = e$ & $\sin\theta = \frac{b}{a}$

$\Rightarrow \tan\frac{\theta}{2} = \frac{a(1-e)}{b} \Rightarrow \tan\frac{\alpha}{2} = \frac{b(1+e)}{a(1-e)^2}$, hence



$$\therefore PN = |b\sin\alpha| = \left|\frac{\frac{2b^2(1+e)}{a(1-e)^2}}{1+\frac{b^2(1+e)^2}{a^2(1-e)^4}}\right|$$

$$= \left|\frac{2ab^2(1-e)^2(1+e)}{a^2(1-e)^4 + b^2(1+e)^2}\right| = 2a\frac{(1-e)^2(1+e)^2}{(1-e)^3+(1+e)^3}$$

$$= a\frac{(1-e^2)^2}{1+3e^2}$$

Q.73 Let P be $(a\cos\theta, b\sin\theta)$

Tangent to the ellipse at P : $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

Length of perpendicular on this tangent from S(ae, 0) will be

$$p = \frac{ab - abe\cos\theta}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

Also $\ell(SP) = a(1 - e\cos\theta)$.

Now $\frac{b^2}{p^2} = \frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2(1-e\cos\theta)^2}$

$$\Rightarrow \frac{b^2}{p^2} = \frac{b^2\cos^2\theta + a^2 - a^2\cos^2\theta}{a^2(1-e\cos\theta)^2}$$

$$\Rightarrow \frac{b^2}{p^2} = \frac{a^2 - a^2e^2\cos^2\theta}{a^2(1-e\cos\theta)^2}$$

$$\Rightarrow \frac{b^2}{p^2} = \frac{1+e\cos\theta}{1-e\cos\theta} = \frac{2a}{\ell(SP)} - 1.$$

Q.74 Let the coordinates of P, P', Q, Q' be $(a\cos\alpha, b\sin\alpha), (-a\cos\alpha, -b\sin\alpha), (a\cos\alpha, a\sin\alpha)$ & $(-a\cos\alpha, -a\sin\alpha)$ respectively.

Now The quadrilateral formed by tangents at these points to the respective curves will be a parallelogram as tangents at the extremities of diameters are parallel.

Equation of tangent at P : $\frac{x\cos\alpha}{a} + \frac{y\sin\alpha}{b} = 1 \ldots(i)$

Equation of tangent at P' : $\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=-1$ …(ii)

Equation of tangent at Q : $x\cos\alpha+y\sin\alpha=a$ …(iii)

Equation of tangent at Q' : $x\cos\alpha+y\sin\alpha=-a$ …(iv)

Point of intersection of (i) & (iii) : $\left(\frac{a}{\cos\alpha},0\right)$

Point of intersection of (i) & (iv) : $\left(\frac{a(a+b)}{(b-a)\cos\alpha},-\frac{2ab}{(b-a)\sin\alpha}\right)$

Point of intersection of (ii) & (iv) : $\left(-\frac{a}{\cos\alpha},0\right)$

Required area = $\left\|\begin{matrix}1 & \frac{a}{\cos\alpha} & 0\\ 1 & -\frac{a}{\cos\alpha} & 0\\ 1 & \frac{a(a+b)}{(b-a)\cos\alpha} & -\frac{2ab}{(b-a)\sin\alpha}\end{matrix}\right\|=\frac{4a^2b}{(a-b)\sin\alpha\cos\alpha}$.

Q.75 Normal at P(θ) : $\frac{\sqrt{14}x}{\cos\theta}-\frac{\sqrt{5}y}{\sin\theta}=9$.

As this normal passes through Q(2θ), hence $\frac{14\cos 2\theta}{\cos\theta}-\frac{5\sin 2\theta}{\sin\theta}=9$

$\Rightarrow 18\cos^2\theta-9\cos\theta-14=0\Rightarrow\cos\theta=-\frac{2}{3}$.

Q.76 Slope of normal at P(θ) : $-\frac{a}{b}\tan\theta$

As normal is inclined to x – axis at 45°, hence $\frac{a}{b}\tan\theta=1$

$\Rightarrow\sin\theta=\frac{b}{\sqrt{a^2+b^2}},\cos\theta=\frac{a}{\sqrt{a^2+b^2}}$.

Hence coordinates of P are $\left(\frac{a^2}{\sqrt{a^2+b^2}},\frac{b^2}{\sqrt{a^2+b^2}}\right)$.

Now any point at a distance r on this normal will be $\left(\frac{a^2}{\sqrt{a^2+b^2}}-\frac{r}{\sqrt{2}},\frac{b^2}{\sqrt{a^2+b^2}}-\frac{r}{\sqrt{2}}\right)$

Substituting these coordinates in the equation of the ellipse gives

$$b^2\left(\frac{a^2}{\sqrt{a^2+b^2}}-\frac{r}{\sqrt{2}}\right)^2+a^2\left(\frac{b^2}{\sqrt{a^2+b^2}}-\frac{r}{\sqrt{2}}\right)^2=a^2b^2$$

$$\Rightarrow \frac{\left(a^2+b^2\right)r^2}{2}=2\sqrt{2}\frac{a^2b^2r}{\sqrt{a^2+b^2}}\Rightarrow r^2=\frac{32a^4b^4}{\left(a^2+b^2\right)^3}.$$

Q.77 Chord of contact of the tangents drawn from R(h, k) to the ellipse $\frac{x^2}{c^2}+\frac{y^2}{d^2}=1$ is

$\frac{hx}{c^2}+\frac{ky}{d^2}=1\ldots$(i)

Any tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will be $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1\ldots$(ii)

comparing (i) & (ii) gives $h=\frac{c^2\cos\theta}{a}$ & $k=\frac{d^2\sin\theta}{b}\ldots$(iii)

But the tangents from R are mutually perpendicular,

therefore $h^2+k^2=c^2+d^2\ldots$(iv)

From (iii) we get $\frac{a^2h^2}{c^4}+\frac{b^2k^2}{d^4}=1$

Comparing the above relation with that in (iv) gives

$\frac{a^2}{c^4}=\frac{b^2}{d^4}=\frac{1}{c^2+d^2}\Rightarrow\frac{a^2}{c^2}=\frac{c^2}{c^2+d^2}$ & $\frac{b^2}{d^2}=\frac{d^2}{c^2+d^2}$ or $\frac{a^2}{c^2}+\frac{b^2}{d^2}=1.$

Q.78 Let the tangents be drawn at P(α) & Q(β),

then equations of the tangents will be

$\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=1\ldots$(i)

& $\frac{x\cos\beta}{a}+\frac{y\sin\beta}{b}=1\ldots$(ii)

and point of intersection of these tangents will be $P(x,y)\equiv\left(\frac{a\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}},\frac{b\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}\right)$.

The tangents will meet the x – axis at $\left(\frac{a}{\cos\alpha},0\right)$ & $\left(\frac{a}{\cos\beta},0\right)$

Hence as given $\left|\frac{a}{\cos\alpha}-\frac{a}{\cos\beta}\right|=c$.

$$\Rightarrow\left|\frac{4\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{\cos(\alpha+\beta)+\cos(\alpha-\beta)}\right|=\frac{c}{a}\Rightarrow\left|\frac{2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{\cos^2\frac{\alpha+\beta}{2}-\sin^2\frac{\alpha-\beta}{2}}\right|=\frac{c}{a}$$

Now for the point P we have $\frac{x}{a}=\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}, \frac{y}{b}=\frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}$.

Using $\sin\frac{\alpha-\beta}{2}=\frac{y}{b}\cos\frac{\alpha+\beta}{2}$, we get $\left|\tan\frac{\alpha+\beta}{2}\right|=\frac{c\left(b^2-y^2\right)}{2aby}$

Also $\frac{x}{a}=\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}, \frac{y}{b}=\frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}} \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\tan^2\frac{\alpha+\beta}{2}$

Hence required locus is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{c^2\left(b^2-y^2\right)^2}{4a^2b^2y^2}$.

Q.79 Pair of tangents drawn from $\left(\frac{a^2}{\sqrt{a^2-b^2}}, \sqrt{a^2+b^2}\right)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{a^2}{a^2-b^2}+\frac{a^2+b^2}{b^2}-1\right)=\left(\frac{1}{\sqrt{a^2-b^2}}x+\frac{\sqrt{a^2+b^2}}{b^2}y-1\right)^2 \quad \left\{\text{By } SS_1=T^2\right\}$$

Let these lines meet the ordinate through (ae, 0) at (ae, k_1) & (ae, k_2)

Substitute (ae, k) in the equation of pair of tangents

$$\left(\frac{a^2e^2}{a^2}+\frac{k^2}{b^2}-1\right)\left(\frac{a^2}{a^2-b^2}+\frac{a^2+b^2}{b^2}-1\right)=\left(\frac{ae}{\sqrt{a^2-b^2}}+\frac{k\sqrt{a^2+b^2}}{b^2}-1\right)^2$$

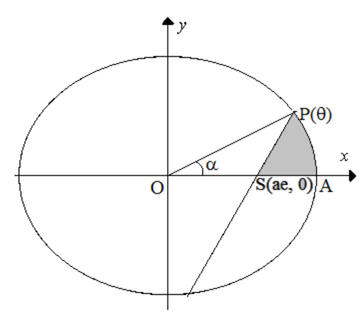
$$\Rightarrow \frac{a^4}{\left(a^2-b^2\right)}\left(e^2+\frac{k^2}{b^2}-1\right)=\frac{k^2\left(a^2+b^2\right)}{b^2}$$

$$\Rightarrow \left(\frac{a^4}{\left(a^2-b^2\right)b^2}-\frac{a^2+b^2}{b^2}\right)k^2=\frac{a^4}{a^2-b^2}-a^2 \Rightarrow k^2=a^2$$

Hence $k_1=a$ & $k_2=-a \Rightarrow k_1-k_2=2a$.

Q.80 Substitute x = r cos α & y = r sin α in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to get

polar equation of the ellipse as $r^2=\frac{a^2b^2}{b^2\cos^2\alpha+a^2\sin^2\alpha}$.



Area of sector OAP = $\frac{1}{2}\int_0^{\alpha}\frac{a^2b^2}{b^2\cos^2\alpha+a^2\sin^2\alpha}d\alpha$

$=\frac{ab}{2}\tan^{-1}\left(\frac{a\tan\alpha}{b}\right)$.

Also $\tan\alpha=\frac{b}{a}\tan\theta$, hence Area of sector OAP = $\frac{ab\theta}{2}$

Area of triangle OPS = $\frac{1}{2} \times ae \times b\sin\theta$.

The required area = area of sector OAP – area of triangle OPS

$= \frac{ab\theta}{2} - \frac{1}{2} \times ae \times b\sin\theta$.