ELLIPSE

EXERCISE 2(C)

Q.1 [04]

Given equations are $\frac{x^2}{16}+\frac{y^2}{9}=1$ & $\frac{x^2}{9}+\frac{y^2}{16}=1$

Subtracting the two equations gives $x=\pm y$

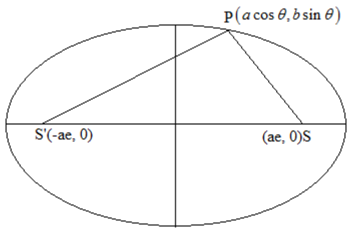
Now $\frac{x^2}{16}+\frac{y^2}{9}=1$ & $x^2=y^2 \Rightarrow x^2=y^2=\frac{144}{25}$

Or $x=\pm\frac{12}{5}$ & $y=\pm\frac{12}{5}$.

Points of intersection $=\left(\pm\frac{12}{5},\pm\frac{12}{5}\right)$

Hence number of points = 4

Q.2 [03]



$SP=a-ae\cos\theta,\ S'P=a+ae\cos\theta$ & $SS'=2ae$.

Now coordinates of in center will be

$$\left(\frac{(a\cos\theta)2ae+ae(a+ae\cos\theta)-ae(a-ae\cos\theta)}{2ae+a+ae\cos\theta+a-ae\cos\theta},\frac{2ae\,b\sin\theta}{2a+2ae}\right)$$

Or $\left(\frac{2a^2\cos\theta+a^2e+a^2e^2\cos\theta-a^2e+a^2e^2\cos\theta}{2a+2ae},\frac{be\sin\theta}{1+e}\right)$

Or $\left(\frac{2a^2e\cos\theta(1+e)}{2a(1+e)},\frac{be\sin\theta}{1-e}\right)$. Now let $h=\frac{2a^2e\cos\theta(1+e)}{2a(1+e)}$ & $k=\frac{be\sin\theta}{1-e}$,

then eliminating θ gives $\frac{h^2}{a^2e^2}+\frac{(1+e)^2k^2}{b^2e^2}=1$

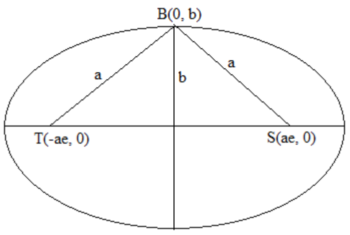
$e=\frac{3}{5} \Rightarrow 1-\frac{a^2}{b^2}=\frac{9}{25}$ or $\frac{b^2}{a^2}=\frac{16}{25}$

Required Locus : $\frac{25x^2}{9a^2}+\frac{64y^2}{9b^2}=1$

Now $\lambda=\sqrt{1-\frac{9b^2}{64a^2}}$ or $\lambda=\sqrt{1-\frac{1}{4}}$

Hence $4\lambda^2 = 3$.

Q.3 [02]



ST = 2*ae* & SB = TB = *a*.

As the triangle is equilateral hence 2*ae* = *a*.

Or *e* = ½.

Hence 4*e* = 2

Q.4 [01]

Let the equation of ellipse be $\frac{(x-1)^2}{64}+\frac{(y+1)^2}{b^2}=1$.

As it passes through (1, 3) hence $\frac{(1-1)^2}{64}+\frac{(3+1)^2}{b^2}=1$ or $b^2=16$.

Now length of latus rectum = $\frac{2b^2}{a}=\frac{32}{8}=4$, therefore $\frac{l}{4}=1$

Q.5 [02]

Let the equation of any tangent be $y=mx+\sqrt{a^2m^2+b^2}$, where $a=67$ & $b=33$

$\Rightarrow y=mx+\sqrt{(67)^2m^2+(33)^2}$.

Also two points on minor axis at a distance of $10\sqrt{34}$ from origin will be

$P\left(0,10\sqrt{34}\right)$ & $Q\left(0,-10\sqrt{34}\right)$.

Sum of squares of perpendiculars on the tangent from P & Q

$$=\frac{\left(10\sqrt{34}+\sqrt{(67)^2m^2+(33)^2}\right)^2+\left(10\sqrt{34}-\sqrt{(67)^2m^2+(33)^2}\right)^2}{(1+m^2)}$$

$$\Rightarrow 4489\lambda=\frac{\left(3400+(67)^2m^2+(33)^2\right)^2}{(1+m^2)} \quad \text{or} \quad 4489\times 2=4489\times\lambda$$

Hence $\lambda=2$.

Q.6 [02]

Given a = 5, b = 4

Hence $e=\sqrt{1-\frac{16}{25}}=\frac{3}{5}$ and $ae = 3$

$SP = \text{radius of circle} = a - ae\cos\theta$

$= 5-3\cos\theta,\ \theta\in\left[0,\frac{\pi}{2}\right].$

$r_{\max} = 5-3 = 2$.

Q.7 [07]

Let equation of common tangent be $y = mx + c$.

For being tangent to ellipse, $c^2 = 25m^2+4$ & for circle $c^2 = 16\left(m^2+1\right)$.

$25m^2+4 = 16m^2+16 \Rightarrow m = \pm\frac{2}{\sqrt{3}}$.

Equation of one of the common tangents : $y=\frac{2}{\sqrt{3}}x+4\sqrt{1+\frac{4}{3}}$

Required area $=\frac{1}{2}\times\frac{4\sqrt{7}}{\sqrt{3}}\times 2\sqrt{7}$

$=\frac{\sqrt{3}L}{4}=7$.

Q.8 [01]

Equation of tangent : $5x\sec\theta-4y\,\text{cosec}\,\theta=9$.

Distance of this line from (0, 0), $l=\frac{9}{\sqrt{25\sec^2\theta+16\cos ec^2\theta}}$.

Now $25\sec^2\theta+16\cos ec^2\theta=41+25\tan^2\theta+16\cot^2\theta$

By A.M.$\geq$ G.M., $25\tan^2\theta+16\cot^2\theta\geq 40$

Hence $25\sec^2\theta+16\cos ec^2\theta\geq 81$

Therefore $\frac{9}{\sqrt{25\sec^2\theta+16\cos ec^2\theta}}\leq 1.$

Q.9 [01]

Let $P(a\cos\theta, b\sin\theta)$ be any general point on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Equation of tangent at this point will be given by $\frac{x\cos\theta}{a}+\frac{b\sin\theta}{a}=1$

at $x=0,\ y=\frac{b}{\sin\theta}$ & at $y=0,\ x=\frac{a}{\cos\theta}$

$$\therefore\ \frac{a^2}{CP^2}+\frac{b^2}{CQ^2}=\frac{a^2}{\left(\frac{a}{\cos\theta}\right)^2}+\frac{b^2}{\left(\frac{b}{\sin\theta}\right)^2}$$

Hence $\frac{a^2}{CP^2}+\frac{b^2}{CQ^2}$ = 1.

Q.10 [04]

The equation of normal at (x',y') is $\frac{x-x'}{x'}a^2=\frac{y-y'}{y'}b^2$

If it passes through (h, k), then $y'^2\{a^2(h-x')+b^2x'\}^2=b^4k^2x'^2$ ....(i)

But $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1\ \ or\ \ y'^2=\frac{b^2}{a^2}(a^2-x'^2)$ ....(ii)

Value of y'^2 from eq.(ii), putting in eq. (i), we get

$$\frac{b^2}{a^2}(a^2-x'^2)\{a^2h+(b^2-a^2)x'^2\}=b^4k^2x'^2$$

$$\Rightarrow\frac{b^2}{a^2}(a^2-x'^2)\{a^4h^2+(b^2-a^2)^2x'^2+2a^2hx'(b^2-x^2)\}=b^4k^2x'^2$$

Arrange above as fourth degree equation in x', then roots of the above equation are x_1,x_2,x_3,x_4. Now

$$x_1.x_2.x_3.x_4=-\frac{2ha^2(a^2-b^2)}{-(a^2-b^2)^2}=\frac{2ha^2}{(a^2-b^2)}$$ ....(iii)

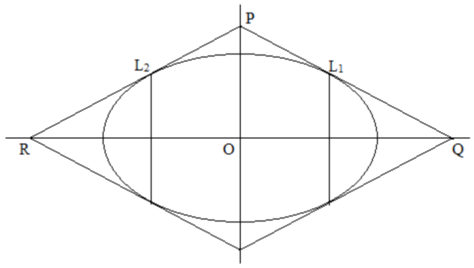
Also $\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=\frac{\sum x_1x_2x_3}{x_1.x_2.x_3.x_4}$

$$=\frac{\dfrac{2a^4h(a^2-b^2)}{-(a^2-b^2)^2}}{\dfrac{a^6h^2}{-(a^2-b^2)^2}}=\frac{2(a^2-b^2)}{a^2h} \qquad \text{.........(iv)}$$

Multiplying (iii) and (iv), we get $(x_1+x_2+x_3+x_4)\times\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4$.

Q.11 [03]

Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. L_1 & L_2 are two of the extremities of latus recta. Now



equation of tangents at $L_1\left(ae,\frac{b^2}{a}\right)$ & $L_2\left(-ae,\frac{b^2}{a}\right)$

will be $\frac{x(ae)}{a^2}+\frac{y\left(\frac{b^2}{a}\right)}{b^2}=1$ & $\frac{x(-ae)}{a^2}+\frac{y\left(\frac{b^2}{a}\right)}{b^2}=1$.

Solving these equations we get P as (0, a).

Also Q & R are $\left(\frac{a}{e},0\right)$ & $\left(-\frac{a}{e},0\right)$.

Now area of triangle PQR $=\frac{1}{2}\times\frac{2a}{e}\times a=\frac{a^2}{e}$

Therefore area of quadrilateral = $\frac{2a^2}{e}$

Now $A=\frac{2a^2}{e}\Rightarrow\frac{A}{9}=3$

Q.12 [05]

The man is running on an ellipse in which $2a = 10$ & $2ae = 8$.

Now $b^2=a^2-a^2e^2$ gives $b = 3$.

Area of ellipse = $\pi ab=15\pi$.

Q.13 [03]

Let the common tangent be $y = mx + c$.

For circle, $c^2 = 16(1+m^2)$ & for ellipse, $c^2 = 25m^2 + 4$

slope of common tangent $= \sqrt{\frac{16-4}{25-16}} = \frac{2}{\sqrt{3}}$

Equation of the tangent : $y = \frac{2x}{\sqrt{3}} + 4\sqrt{1+\frac{4}{3}}$ or $y = \frac{2x}{\sqrt{3}} + \frac{4\sqrt{7}}{\sqrt{3}}$

$x-$intercept $= -2\sqrt{7}$ & $y-$intercept $= \frac{4\sqrt{7}}{\sqrt{3}}$

Reqd. Length $= \sqrt{28 + \frac{112}{3}} = \frac{14}{\sqrt{3}}$.

Q.14 [04]

Equation of chord joining $P(\theta_1) \& Q(\theta_2)$: $\frac{x}{a}\cos\frac{\theta_1+\theta_2}{2} + \frac{y}{b}\sin\frac{\theta_1+\theta_2}{2} = \cos\frac{\theta_1-\theta_2}{2}$

or $\frac{2x}{a}\cos\frac{\theta_1+\theta_2}{2} + \frac{2y}{b}\sin\frac{\theta_1+\theta_2}{2} = 1$. {given that $\theta_1 - \theta_2 = \frac{2\pi}{3}$ }

Let the point of intersection of tangents at $P(\theta_1) \& Q(\theta_2)$ be R(h, k).

Chord of contact of R : $\frac{hx}{a^2} + \frac{ky}{b^2} = 1$.

Comparing the two equations gives $\frac{h}{2a} = \cos\frac{\theta_1+\theta_2}{2}$ & $\frac{k}{2b} = \sin\frac{\theta_1+\theta_2}{2}$

Square and add to get $\frac{h^2}{4a^2} + \frac{k^2}{4b^2} = 1$.

Required locus is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 4$.

Q.15 [04]

Given : $C \equiv (2,-3)$, $S \equiv (3,-3)$, $A \equiv (4,-3)$

Now a = CA = 2 & ae = CS = 1.

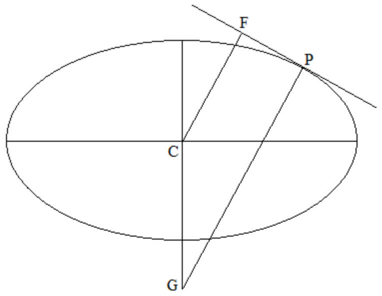
Hence $b = a\sqrt{1-e^2} = \sqrt{3}$.

Equation of ellipse : $\frac{(x-2)^2}{4} + \frac{(y+3)^2}{3} = 1$ or $3x^2 + 4y^2 - 12x + 24y + 36 = 0$.

Q.16 [01]

Standard result : $\frac{1}{e_1^2}+\frac{1}{e_2^2}=1$.

Q.17 [07]



Given ellipse is $\frac{x^2}{49}+\frac{y^2}{25}=1$

Let any point be $P(7\cos\theta, 5\sin\theta)$

Equation of tangent P : $\frac{x\cos\theta}{7}+\frac{y\sin\theta}{5}=1$

Perpendicular from center, $CF = \frac{1}{\sqrt{\frac{\cos^2\theta}{49}}+\sqrt{\frac{\sin^2\theta}{25}}}$ or $\frac{35}{\sqrt{25\cos^2\theta+49\sin^2\theta}}$

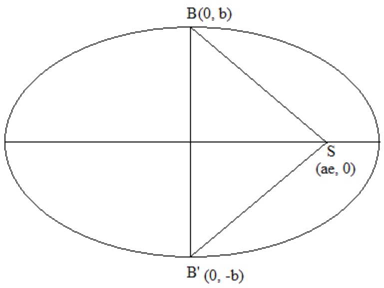
Equation of normal at Q : $7x\sec\theta-5y\,\mathrm{cosec}\,\theta=24$

Now at $x=0$, $y=-\frac{24}{5}\sin\theta$ Hence $G\equiv\left(0,-\frac{24}{5}\sin\theta\right)$

Hence $PG=\sqrt{49\cos^2\theta+\left(5\sin\theta+\frac{24}{5}\sin\theta\right)^2}$ or $\frac{7}{5}\sqrt{25\cos^2\theta+49\sin^2\theta}$

$\therefore \sqrt{CF.PG} = 7$.

Q.18 [01]



Given $m_{SB}\times m_{SB'} = -1$

$\Rightarrow\left(\frac{b-0}{0-a}\right).\left(\frac{a+b}{ae-0}\right)=-1$

$\Rightarrow\frac{b^2}{a^2e^2}=-1$

$\frac{b^2}{a^2}=\frac{1}{2}\Rightarrow e=\sqrt{1-\frac{b^2}{a^2}} = \frac{1}{\sqrt{2}}$.

Q.19 [09]

If the equation of tangent to the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$, is $y = mx + c$, then $c^2 = 25m^2 + 9$

Now given equation is $y = 2x + \lambda$ so $m = 2$ & $c = \lambda$.

$\therefore\ \lambda^2 = 25\times4+9$ or $\lambda^2 - 100 = 9$.

Q.20 [04]

Standard fact : In general from any point four normals can be drawn to an ellipse.

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