ELLIPSE

EXERCISE 1(C)

Q.1 (A)

Tangents at the end points of latus rectum intersect on minor axis and directrices.
Vertices of the quadrilateral will be (0, a), (0, −a), (a/e, 0) & (−a/e, 0).

Area = $\frac{1}{2}\times 2a\times\frac{2a}{e}=\frac{2a^2}{e}$

Now $\frac{x^2}{2}+\frac{y^2}{1}=1$, gives $a^2=2, b^2=1\rightarrow e=\frac{1}{\sqrt{2}}$, hence

Area = $4\sqrt{2}$.

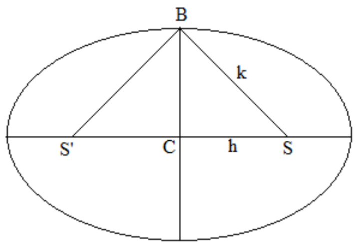
Q.2 (A)

$\frac{x^2}{10-a}+\frac{y^2}{4-a}=1$ will be equation of an ellipse if

$10-a>0$ i.e. $a<10$ & $4-a>0$ i.e. $a<4$.

Hence correct answer is $a<4$.

Q.3 (B)



Given $b^2=CB^2=k^2-h^2$ & $a=BS=k$.

Equation of ellipse : $\frac{x^2}{k^2}+\frac{y^2}{k^2-h^2}=1$.

Q.4 (B)

Let the tangent be $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$.

These tangents meet the coordinate axes at $P\left(\frac{a}{\cos\theta},0\right)$ & $Q\left(0,\frac{b}{\sin\theta}\right)$.

Midpoint of PQ : $x=\frac{a}{2\cos\theta}, y=\frac{b}{2\sin\theta}$.

Eliminating θ, gives $\frac{a^2}{x^2}+\frac{b^2}{y^2}=4$.

Q.5 (B)

BS = BT = a & ST = 2ae.

As the triangle is equilateral hence 2ae = a or e = ½.

Q.6 (B)

The two given points are $P(-ae, 0)$ & $Q(0, ae)$

Geometrical property of any ellipse : product of perpendiculars from the two foci on any tangent is equal to square of semi minor axis.

Q.7 (A)

P(0, 0), S(5, 12) & S'(24, 7).

Now SP + S'P = 2a $\Rightarrow 2a = \sqrt{5^2 + 12^2} + \sqrt{24^2 + 7^2}$ or $2a = 38$.

Also SS' = 2ae $\Rightarrow 2ae = \sqrt{(5-24)^2 + (12-7)^2}$ or $2ae = \sqrt{386}$.

Hence e = $\dfrac{\sqrt{386}}{38}$.

Q.8 (D)

Let the common tangent be y = mx + c.

For being a tangent to $x^2 + y^2 = 4 \rightarrow c^2 = 4(m^2 + 1)$.

For being a tangent to $x^2 + \dfrac{y^2}{2} = 1 \rightarrow c^2 = m^2 + 2$.

Hence $4m^2 + 4 = m^2 + 2$ or $3m^2 = -2$.

There are no common tangents.

Alternately

Both the semi major axis and semi minor axis are of length less than the radius of circle hence the ellipse is completely enclosed in the circle. There cannot be any common tangents.

Q.9 (C)

Coordinates of end points of latus rectum $\left(\pm ae, \pm\dfrac{b^2}{a}\right)$.

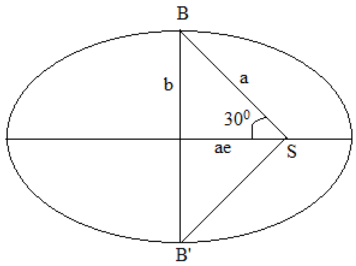
Comparing with $(a\cos\theta, b\sin\theta)$ gives $\cos\theta = \pm e$ & $\sin\theta = \pm\dfrac{b}{a}$.

Hence $\tan\theta = \pm\frac{b}{ae}$ or $\theta = \tan^{-1}\left(\pm\frac{b}{ae}\right)$.

Q.10 (B)

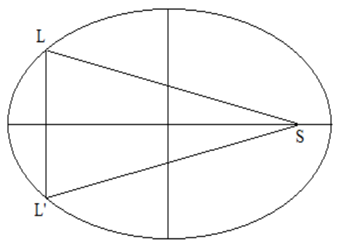
Standard definition of an ellipse.

Q.11



$$\frac{ae}{a} = \tan 30^0 \text{ or } e = \frac{1}{\sqrt{3}}$$

Q.12 (B)



$L\left(-ae, \frac{b^2}{a}\right), L'\left(-ae, -\frac{b^2}{a}\right) \& S(ae, 0)$

Now $\tan 30^o = \frac{\frac{b^2}{a}}{2ae} \Rightarrow \frac{b^2}{a^2} = \frac{2e}{\sqrt{3}}$

$\Rightarrow 1 - e^2 = \frac{2e}{\sqrt{3}}$ or $\sqrt{3}e^2 + 2e - \sqrt{3} = 0$, hence $e = \frac{1}{\sqrt{3}}$.

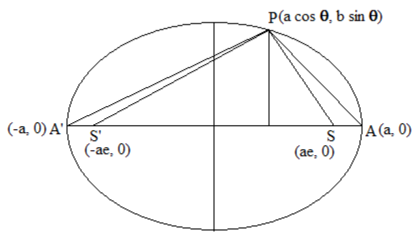
Q.13 (A)

$\frac{x^2}{\cot^2\alpha} + \frac{y^2}{\cos^2\alpha} = 1 \to a^2 = \cot^2\alpha, b^2 = \cos^2\alpha$

Now L.R. $= \frac{2b^2}{a^2} \Rightarrow \frac{2\cos^2\alpha}{\cot\alpha} = \frac{1}{2}$ or $\sin 2\alpha = \frac{1}{2}$.

Hence $\alpha = \frac{\pi}{12}$.

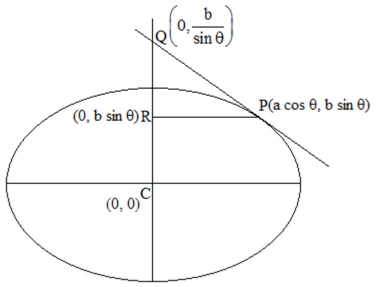
Q.14 (C)



Now $A_{SPS'} = \frac{1}{2} \times 2ae \times b\sin\theta$ & $A_{APA'} = \frac{1}{2} \times 2a \times b\sin\theta$.

Hence $\frac{A_{SPS'}}{A_{APA'}} = e$.

Q.15 (A)



$$CQ \times CR = b\sin\theta \times \frac{b}{\sin\theta} \text{ i.e. } b^2$$

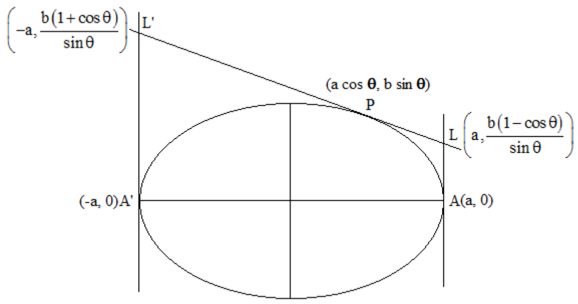
Q.16 (C)

Circle on SS' as diameter touches the ellipse hence ae = b.

$$\Rightarrow \frac{b^2}{a^2} = e^2 \text{ or } 1 - e^2 = e^2$$

$$\Rightarrow e = \frac{1}{\sqrt{2}}.$$

Q.17 (D)



$$AL \times A'L' = \frac{b(1-\cos\theta)}{\sin\theta} \times \frac{b(1+\cos\theta)}{\sin\theta} \text{ i.e. } b^2.$$

Q.18 (D)

Let equation of tangent be $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

Hence coordinates of P & Q are (a sec θ, 0) & (0, b cosec θ).

$$\Rightarrow |CP|^2 = a^2\sec^2\theta, \quad |CQ|^2 = b^2\operatorname{cosec}^2\theta$$

$$\text{or } \frac{a^2}{|CP|^2} + \frac{b^2}{|CQ|^2} = \cos^2\theta + \sin^2\theta = 1.$$

Q.19 (C)

$$x = ae \Rightarrow x^2 = a^2 - b^2$$

$$y = \frac{b^2}{a} \Rightarrow x^2 = a^2 - ay.$$

Q.20 (A)

The circle lies completely inside the ellipse hence length of common chord = 0.

Q.21 (A)

As intercepts are equal hence slope = 1 or –1.

Equations of tangents : $y \pm x = \pm\sqrt{a^2 + b^2}$

Length of intercept, $\ell = \sqrt{a^2 + b^2}$.

Q.22 (D)

CS = ae = $\sqrt{7}$, Major axis = 2a = 8.

CS : Major axis = $\sqrt{7}$: 8.

Q.23 (A)

Let P be $(a\cos\theta, b\sin\theta)$.

Now area of triangle APA' = $\frac{1}{2} \times 2a \times b\sin\theta$.

Clearly greatest area = ab, when $\sin\theta = 1$.

Q.24

Parametric coordinates of any point at a distance r from (ae, 0) on a line with slope tan θ and passing through (ae, 0) will be (ae + r cos θ, r sin θ).

Substituting these coordinates in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ gives

$$\frac{(ae + r\cos\theta)^2}{a^2} + \frac{(r\sin\theta)^2}{b^2} = 1 \text{ or } \left(b^2\cos^2\theta + a^2\sin^2\theta\right)r^2 + \left(2ab^2 e\cos\theta\right)r + a^2b^2e^2 - a^2b^2 = 0$$

Now SP & –SQ are the roots of this equation, thus

$$SP \times (-SQ) = \frac{a^2b^2\left(e^2 - 1\right)}{b^2\cos^2\theta + a^2\sin^2\theta} \text{ \& } SP - (-SQ) = \frac{\sqrt{4a^2b^4e^2\cos^2\theta - 4a^2b^2\left(e^2 - 1\right)\left(b^2\cos^2\theta + a^2\sin^2\theta\right)}}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$\text{Or } SP \times SQ = \frac{b^4}{b^2\cos^2\theta + a^2\sin^2\theta} \text{ \& } SP + SQ = \frac{2ab^2}{b^2\cos^2\theta + a^2\sin^2\theta}$$

Now $\frac{2 \times SP \times SQ}{SP + SQ} = \frac{b^2}{a}$.

Q.25 (B)

Let the ellipse be $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$. Now

$A^2 = a^2 + 1$ & $B^2 = a^2 + 2 \rightarrow e^2 = \dfrac{B^2 - A^2}{B^2} \Rightarrow \dfrac{1}{a^2+2} = \dfrac{1}{6} \Rightarrow a = 2$

$\Rightarrow A^2 = 5,\ B^2 = 6$

$\Rightarrow \text{L.R.} = \dfrac{2A^2}{B} = \dfrac{10}{\sqrt{6}}$

Q.26 (A)

Let end points of a double ordinate be $P(a\cos\theta, b\sin\theta)$ & $Q(a\cos\theta, -b\sin\theta)$.

The point which divides PQ in 1:2 internally - $R\left(a\cos\theta, \dfrac{b\sin\theta}{3}\right)$

Eliminating θ between x & y coordinates of R gives required locus as $\dfrac{x^2}{a^2} + \dfrac{9y^2}{b^2} = 1$.

Q.27 (B)

Slope is $-\dfrac{4}{3}$, hence equation of tangent is $y = -\dfrac{4}{3}x + \sqrt{18 \times \dfrac{16}{9} + 32}$ or $4x + 3y = 24$

x – intercept = 6, y – intercept = 8.

Area of triangle ABC = 24.

Q.28 (B)

Product of perpendiculars from the foci on any tangent is = $(\text{semi minor axis})^2$

Hence $S_1F_1 \times S_2F_2 = 3$.

Q.29 (A)

Let the common tangent be y = mx + c.

For $y^2 = 4x : c = \dfrac{1}{m}$.

For $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1 : c^2 = 4m^2 + 3$.

Now $\dfrac{1}{m} = 4m^2 + 3 \Rightarrow 4m^4 + 3m^2 - 1 = 0$.

Thus $m = \pm\dfrac{1}{2}$.

Now equations of common tangents will be $2y = x + 4$ & $2y = -x - 4$.

Q.30 (D)

Ellipse meets 2x – 3y = 6 on x – axis hence a = 3.

Ellipse meets 4x + 5y = 20 on y – axis hence b = 4.

Now $e = \sqrt{1 - \frac{a^2}{b^2}} \Rightarrow e = \frac{\sqrt{7}}{4}$.

Q.31 (A)

Let the common tangent be y = mx + c.

For being a tangent to $x^2 + y^2 = r^2 \rightarrow c^2 = r^2\left(m^2 + 1\right)$.

For being a tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow c^2 = a^2m^2 + b^2$.

Hence $r^2m^2 + r^2 = a^2m^2 + b^2$ or $m^2 = \frac{r^2 - b^2}{a^2 - r^2}$.

Inclination of common chord with x – axis = $\pm\tan^{-1}\sqrt{\frac{r^2 - b^2}{a^2 - r^2}}$.

Q.32 (A)(C)(D)

For four distinct intersection points, $a^2 > 1$ i.e. $a < -1$ or $a > 1$.

Now $b^2 - 10b + 25 > 1$ gives $b^2 - 10b + 24 > 0$ or $b \in R - [4, 6]$.

Q.33 (B)

$e_1^{\,2} = \frac{a_1^{\,2} - b_1^{\,2}}{a_1^{\,2}}$ & $e_2^{\,2} = \frac{a_2^{\,2} + b_2^{\,2}}{a_2^{\,2}}$

Let $a_1e_1 = a_2e_2 = h$ & $b_1 = b_2 = b$

then $a_1^{\,2} = h^2 + b^2$ & $a_2^{\,2} = h^2 - b^2$

hence $e_1^{\,2} = \frac{h^2}{h^2 + b^2}$ & $e_2^{\,2} = \frac{h^2}{h^2 - b^2}$

$\Rightarrow \frac{1}{e_1^{\,2}} + \frac{1}{e_2^{\,2}} = \frac{h^2 + b^2}{h^2} + \frac{h^2 - b^2}{h^2}$

$\Rightarrow \frac{1}{e_1^{\,2}} + \frac{1}{e_2^{\,2}} = 2.$

Q.34 (A)

Let focus of the parabola be (ae, 0) and directrics be x = -ae.

Now Latus rectum of parabola = 2(distance between focus & directrics) i.e. 4ae.

Latus rectum of the ellipse = $\frac{2b^2}{a}$.

$\frac{2b^2}{a} = 4ae \Rightarrow \frac{b^2}{a^2} = 2e$ or $e^2 + 2e - 1 = 0$.

Hence $e = \sqrt{2} - 1$.