Solutions

PARABOLA

Ex. 3

Q.1 (A)

$$\sqrt{x^2+y^2}=\frac{|x+y-4|}{\sqrt{2}} \Rightarrow x^2-2xy+y^2+8x+8y-16=0.$$

Q.2 (A)

$$\frac{1}{SP}+\frac{1}{SQ}=\frac{1}{a} \Rightarrow 4a=\frac{24}{5}.$$

Q.3 (A)

Latus rectum = twice the distance of the directrics from the focus.
$\Rightarrow$ Latus rectum = 2.

Q.4 (D)

$$y^2+4y+4x+2=0 \Rightarrow (y+2)^2=-4\left(x-\frac{1}{2}\right).$$

Hence equation of directrics is $x=\frac{3}{2}$.

Q.5 (A)

Standard fact : $h^2 = ab$.

Q.6 (A)

Given equation is standard equation of a parabola in terms of its focus(2, 3) and directrics (3x + 4y – 2 = 0).

Q.7 (B)

Standard fact : From any point (h, 0) three distinct normals can be drawn to the parabola $y^2 = 4ax$ if h > 2a and only one normal (axis) can be drawn if h < 2a.

Q.8 (C)

Standard fact : If tangent and normal at any point P on a parabola meet the axis at T & G and S is the focus than ST = SP = SG.

Q.9 (D)

Standard fact : Segment of any tangent, to a parabola, intercepted between the directrics and the point of tangency subtends a right angle at the focus.

Q.10 (A)

Point on $y = x^2$ which is at least distance from $y = 2x - 4$ will be the point where normal to the parabola is perpendicular to this line.

Now equation of normal having slope m to $x^2 = 4ay$: $y = mx + 2a + \frac{a}{m^2}$, foot of normal being

$\left(-\frac{2a}{m}, \frac{a}{m^2}\right)$.

Here $m = -\frac{1}{2}$ & $a = \frac{1}{4}$, hence the required point is (1, 1).

Q.11 (C)

Equation of tangent at the vertex : $x - y + 1 = 0$.

Focus : (0, 0) & Focal distance of the vertex : $a = \frac{1}{\sqrt{2}}$

Equation of directrics : $x - y + 2 = 0$. {Line parallel to the tangent at the vertex at a distance 'a' from it}

Now equation of the parabola : $\sqrt{x^2 + y^2} = \left|\frac{x - y + 2}{\sqrt{2}}\right|$

or $x^2 + 2xy + y^2 - 4x + 4y - 4 = 0$.

Q.12 (D)

Standard fact : Length of subnormal at any point on $y^2 = 4ax$ is 2a.

Q.13 (C)

The two parabolas touch each other at (0, -1) hence angle of intersection is 0.

Q.14 (B)

Let the common tangent be $y = mx + c$, then

for being a tangent to $y^2 = 8ax$, $c = \frac{2a}{m}$ and

for being a tangent to $x^2 + y^2 = 2a^2$, $c^2 = 2a^2(1 + m^2)$.

Hence $\frac{4a^2}{m^2} = 2a^2(1 + m^2)$ or $m^4 + m^2 - 2 = 0$.

Slopes of common tangents are 1 & –1.

∴ Equations of common tangents are $y = \pm x \pm 2a$.

Q.15

Any tangent having slope m to $y^2 = 4x$ is $y = mx + \frac{1}{m}$.

As it is drawn through (–2, –1) hence $-1 = -2m + \frac{1}{m}$ or $2m^2 - m - 1 = 0$.

Now $m_1 + m_2 = \frac{1}{2}$ & $m_1 m_2 = -\frac{1}{2} \Rightarrow m_1 - m_2 = \frac{3}{2}$.

Hence angle between tangents will be given by $\tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2}$ or $\tan\alpha = 3$.

Q.16

Directrics of $y^2 = k\left(x - \frac{8}{k}\right)$ will be $x - \frac{8}{k} = -\frac{k}{4}$.

Given that the directrics is x = 1, hence $\frac{8}{k} - \frac{k}{4} = 1$ or $k^2 + 4k - 32 = 0$.

Hence k = 4 & – 8.

Q.17 (A)

Any tangent of slope m to $(x - 6)^2 + y^2 = 2$ will be
$y = m(x-6) \pm \sqrt{2}\sqrt{1+m^2}$.
Also focus of $y^2 = 16x$ is (4, 0).
As the tangent passes through the focus, hence by substituting coordinates of focus in the equation of the tangent we get $m^2 = 1$.

Q.18 (C)

Let the moving point on parabola be $P(at^2, 2at)$.
Also let midpoint of P & (a, 0) be Q(h, k).

Now $h = \frac{a + at^2}{2}, k = at$.

Eliminating t between h & k gives $k^2 = a(2h - a)$.

Locus of Q will be $y^2 = 2a\left(x - \frac{a}{2}\right)$.

Directrics of this parabola is x = 0.

Q.19 (B)

Length of double ordinate = 8a
$\therefore$ Coordinates of extremities of this double ordinate will be P(4a, 4a) & Q(–4a, 4a).

Now slope of OP = 1 & slope of OQ = –1, hence angle between OP & OQ = $\frac{\pi}{2}$.

Q.20 (A)

$y^2 - 2y - 4x + 5 = 0 \Rightarrow (y-1)^2 = 4(x-1)$.
Now axis of the parabola : y = 1.
Vertex : (1, 1) & a = 1.
Now from any point on the axis three normal can be drawn if it is at a distance more than 2 from the vertex. Hence such a point will be (x, 1), x > 3.

Q.21 (A)

Slope of $y = \sqrt{3}x - 3,$ is $\sqrt{3}$, hence any point on this line at a distance r from $P\left(\sqrt{3}, 0\right)$ will be $\left(\sqrt{3} + \frac{r}{2}, \frac{\sqrt{3}\,r}{2}\right)$. Substituting these coordinates in the equation of the parabola gives

$\frac{3r^2}{4} = \sqrt{3} + \frac{r}{2} + 2$ or $3r^2 + 2r - 4\left(\sqrt{3} + 2\right) = 0$.

Now PA.PB will be positive product of roots of this equation.

$\therefore$ PA.PB = $\frac{4\left(\sqrt{3}+2\right)}{3}$.

Q.22 (C)

Let the length of tangent from the origin to the circle passing through A & B be *l*.
As O, A & B are collinear hence l^2 = OA.OB

Now OA & OB will be the roots of $ax^2 + bx + c = 0$ $\therefore$ $l^2 = \frac{c}{a}$.

Q.23 (D)

As normals at the given points are concurrent hence p + q + r = 0.
Clearly 1 satisfies both the equations.

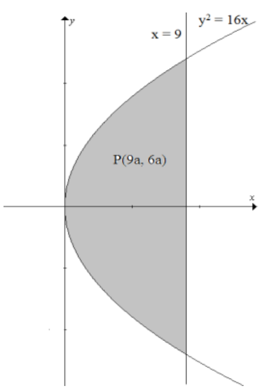
Q.24 (B)

Coordinates of R will be $\left(1, \frac{3\lambda+1}{\lambda+1}\right)$.

If it lies inside $y^2 = 4x$, then $\left(\frac{3\lambda+1}{\lambda+1}\right)^2 < 4$ or $-2 < \frac{3\lambda+1}{\lambda+1} < 2$.

$\therefore$ $-\frac{3}{5} < \lambda < 1$.

Q.25 (A)



Clearly $9a < 9$ & $(6a)^2 < 16(9a)$.

$\Rightarrow a < 1$ & $0 < a < 4$.

Hence $a \in (0, 1)$.

Q.26 (C)

For $(1,-2)$, $x^2+2xy+y^2+2x+3y+1<0$, hence this point lies inside the curve.
No tangent can be drawn from (1, −2).

Q.27 (A)

Let the common tangent be y = mx + c, then

for being a tangent to $y^2 = 8x$, $c=\frac{2}{m}$ and

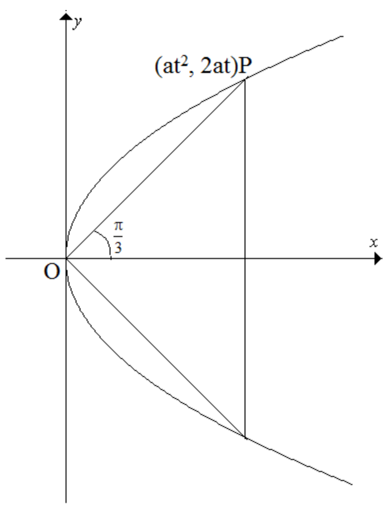
for being a tangent to $x^2 + y^2 = 2$, $c^2=2\left(1+m^2\right)$.

Hence $\frac{4}{m^2}=2\left(1+m^2\right)$ or $m^4+m^2-2=0$.

Slopes of common tangents are 1 & −1.

$\therefore$ There are two mutually perpendicular common tangents.

Q.28 (D)



Slope of OP = $\tan\frac{\pi}{3} \Rightarrow \frac{2}{t}=\frac{1}{\sqrt{3}}$ or $t=2\sqrt{3}$

Hence coordinates of P will be $\left(12a, 4\sqrt{3}a\right)$.

Now length of OP = $\sqrt{144a^2+48a^2}$ i.e. $8\sqrt{3}a$.

Q.29 (A)

If the circle is touching parabola externally, then its center $(-\lambda, 0)$ must lie outside this parabola.

$\Rightarrow 0 > -4\lambda$ or $\lambda > 0$.

Q.30 (A)

$t^2 = x - 1$ & $t = \frac{y-1}{2}$ gives $(y-1)^2 = 4(x-1)$.

Equation of directrics will be x – 1 = –1 i.e. x = 0.

Q.31 (B)

1. $y^2 = 4a(x + a) \Rightarrow$ Focus will be (0, 0) but focus at the origin $\not\Rightarrow y^2 = 4a(x + a)$.

2. $lx+my+n=0 \Rightarrow y=-\frac{l}{m}x-\frac{n}{m}$.

As its touching $y^2 = 4ax$, hence $\frac{n}{m} = \frac{a}{l/m}$ or $nl = am^2$.

Q.32 (D)

$$y = \frac{a^3x^2}{3} + \frac{a^2x}{2} \Rightarrow \left(x^2 + \frac{3}{4a}\right)^2 = \frac{3}{a^3}\left(y + \frac{3a}{16}\right).$$

Now vertex will be $h = -\frac{3}{4a}$ & $k = -\frac{3a}{16}$.

By eliminating a we get $hk = \frac{9}{64}$.

Required locus is $64xy = 9$.

Q.33 (B)

Standard fact : If TP & TQ are tangents & S is the focus, then $SP \times SQ = ST^2$.

Q.34 (D)

If normals at $P(at_1^2, 2at_1)$ & $Q(at_2^2, 2at_2)$ meet at a point R on the parabola, then $t_1t_2 = 2$.
Hence product of ordinates of P & Q will be $8a^2$.

Q.35 (A)

Slope of chord having one end at (0, 0) & other at $P(2t, t^2)$, is $\frac{t}{2}$ and length is $l = \sqrt{4t^2 + t^4}$.

Now given $\frac{t}{2} = \cot\alpha$ $\therefore$ $l = \sqrt{16\cot^2\alpha + 16\cot^4\alpha}$ or $4\cot\alpha\,\text{cosec}\,\alpha$.

Q.36 (C)

Area of triangle formed by $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ & $R(at_3^2, 2at_3)$ will be

$$A = \frac{1}{2}\left\|\begin{matrix} 1 & at_1^2 & 2at_1 \\ 1 & at_2^2 & 2at_1 \\ 1 & at_3^2 & 2at_3 \end{matrix}\right\| \text{ i.e. } a^2\left|(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)\right|.$$

Now $t_1 = \frac{y_1}{2a}, t_2 = \frac{y_2}{2a}$ & $t_3 = \frac{y_3}{2a}$, hence

$$A = \frac{1}{8a}\left|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)\right|.$$

Q.37 (B)

Tangent to $y^2 = 4x$ at (1, 2) is $2y = 2(x + 1)$ or $x - y + 1 = 0$.
Now image of any point P(h, k) will be given by

$$\frac{x-h}{2} = \frac{y-k}{-2} = -\frac{h-k+1}{2} \quad \text{i.e. } x = k-1 \;\&\; y = h+1.$$

If Q(x, y) lies on the given parabola, then P(h, k) will lie on the reflection of this parabola.

Substituting coordinates of Q in y2 = 4x gives $(h + 1)^2 = 4(k – 1)$
Hence reflection of this parabola will be locus of P i.e. $(x + 1)^2 = 4(y – 1)$.

Q.38 (C)

Let $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ & $R(at_3^2, 2at_3)$ be feet of normals concurrent at (h, k).
Now $t_1 + t_2 + t_3 = 0$, hence ordinate centroid of ΔPQR will be 0.
Centroid lies on x – axis.

Q.39 (C)

Standard fact : From any point (c, 0) three distinct normals can be drawn to the parabola $y^2 = 4ax$ if c > 2a. Hence for $y^2 = x$, a = 1/4, and c > ½.

Q.40 (C)

Length of Latus rectum = Twice the normal distance between focus & directrics.
∴ Latus rectum = 10.

Q.41 (A)

For $y^2 = 4ax$ the directrics is x = – a,

hence for $(y + 3)^2 = 2(x + 2)$ directrics will be $x + 2 = -\frac{1}{2}$ or 2x + 5 = 0.

Q.42 (A)

Standard fact : Angle between tangents at extremities of any focal chord is a right angle.

Q.43 (D)

Any line, except x – axis, passing through (2, 0) will meet the parabola in two points as this point lies inside the parabola.
Hence range of values of m is $(-\infty, \infty) - \{0\}$.

Q.44

Let A be $(at_1^2, 2at_1)$ & B be $(at_2^2, 2at_2)$, then equation of circle will be

$$\left(x - at_1^2\right)\left(x - at_2^2\right) + \left(y - 2at_1\right)\left(y - 2at_2\right) = 0$$

or $x^2 + y^2 - a\left(t_1^2 + t_2^2\right)x - 2a\left(t_1 + t_2\right)y + a^2\left(t_1^2 t_2^2 + 4t_1 t_2\right) = 0$

Now radius $r = a\left(t_1 + t_2\right)$ as the circle is touching x – axis.

Also slope of chord AB = $\frac{2}{t_1 + t_2}$ ∴ slope of AB = $\frac{2a}{r}$.

Q.45 (A)

For PQ to be a normal chord, $t_2 = -t_1 - \frac{2}{t_1}$ and

for PQ to subtend right angle at the origin, $t_1 t_2 = -4$.

From the two relations we get $t_1^2 = 2$.

Q.46 (B)

Standard fact : Any line parallel to the axis of a parabola passes through its focus after getting reflected from the parabola.
Now focus of $(y + 3)^2 = 4(x + 1)$ is $(1 - 1, 0 - 3)$ or $(0, -3)$.

Q.47 (C)

Let the moving point on parabola be $P(at^2, 2at)$.
Also let midpoint of P & (a, 0) be Q(h, k).

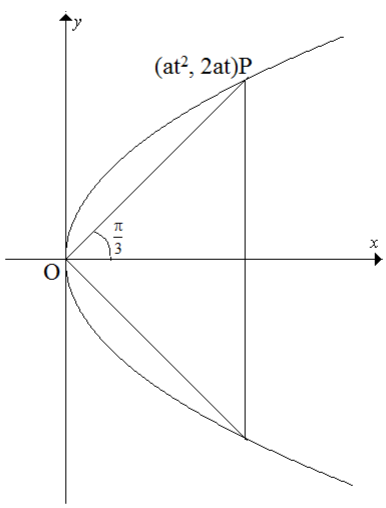
Now $h = \dfrac{a + at^2}{2}, k = at$.

Eliminating t between h & k gives $k^2 = a(2h - a)$.

Locus of Q will be $y^2 = 2a\left(x - \dfrac{a}{2}\right)$.

Vertex of this parabola is $\left(\dfrac{a}{2}, 0\right)$.

Q.48 (B)



Slope of OP = $\tan\dfrac{\pi}{3} \Rightarrow \dfrac{2}{t} = \dfrac{1}{\sqrt{3}}$ or $t = 2\sqrt{3}$

Hence coordinates of P will be $\left(12a, 4\sqrt{3}a\right)$.

Length of OP = $\sqrt{144a^2 + 48a^2}$ i.e. $8\sqrt{3}a$.

Now area of an equilateral triangle = $\dfrac{\sqrt{3}}{4}(\text{side})^2$

i.e. $48\sqrt{3}a^2$.

Q.49 (B)

Latus rectum is parallel to y – axis and focus is at (3, –1).
Length of Latus rectum = 2, hence a = ½.
Now foot of directrics will be at a distance 1 from focus and along a line parallel to x – axis, passing through (3, –1).
Hence foot of directrics is (2, –1) or (4, –1).
Equation of directrics is x = 2 or x = 4.

Q.50 (C)

For P, x = y gives the point on $y^2 = 4ax$ as P(4a, 4a) as normal at origin doesn't meet the parabola again so (0,0) can't be P.
Now $(at^2, 2at) \equiv (4a, 4a)$ gives t = 2.

$\therefore$ For $Q(at_1^2, 2at_1)$, $t_1 = -t - \frac{2}{t}$ gives $t_1 = -3$ and hence Q is (9a, −6a).

Now slope of PF = 4/3 & that of QF = −3/4.

As product of slopes is −1 so $\angle PFQ = \frac{\pi}{2}$.