Solutions

PARABOLA

Ex. 2

Q.1 (A)

Required equation will be

$$\sqrt{(x-5)^2+(y-3)^2}=\frac{|3x-4y+1|}{5}$$

or $(4x+3y)^2-256x-142y+849=0.$

Q.2 (D)

Distance between tangent at the vertex & latus rectum $=\dfrac{12-8}{\sqrt{2}},$

hence Latus Rectum $=8\sqrt{2}.$

Q.3 (C)

Any parabola symmatric about $Y-$Axis & having vertex at the origin will be $x^2=4ay.$

As this parabola passes through $(6,-3)$ hence $a=-3.$

Equation of the parabola is $x^2=-12y.$

Q.4 (C)

$y=2x^2+x$ gives $\left(x+\dfrac{1}{4}\right)^2=4\times\dfrac{1}{8}\left(y+\dfrac{1}{8}\right).$

Now focus of $x^2=4ay$ is $(0,a)$ hence that of the given parabola will be

$\left(0-\dfrac{1}{4},\dfrac{1}{8}-\dfrac{1}{8}\right)$ i.e. $\left(-\dfrac{1}{4},0\right).$

Q.5 (C)

$x^2-4x-8y+12=0$ gives $(x-2)^2=8(y-1).$

Hence length of latus rectum is 8.

Q.6 (D)

$y^2=18x$ & $y=3x \Rightarrow (x,y)\equiv(2,6)$ or $(0,0).$

Q.7 (C)

As the line joining focus and vertex is parallel to X – Axis hence required parabola is in standard form and as focus is on right of vertex so the parabola is opening on right hand side.

Now distance of focus from vertex = 2

$\therefore$ Latus rectum = 8.
Required equation is $(y-1)^2 = 8(x-1)$.

Q.8 (B)

Let a point P on the parabola be $(at^2, 2at)$.
Also the focus S is $(a, 0)$.
Now midpoint of SP will be $x = at^2/2$ & $y = at$.
Eliminating 't' gives the locus of midpoint of SP as $y^2 = 2ax$.
Directrics of this parabola is $x = -a/2$.

Q.9 (D)

Equation of tangent to $y^2 = 9x$ in slope form is $y = mx + \dfrac{9}{4m}$.
As this tangent passes through (4, 10) hence substitute these coordinates in above equation of tangent and simplify to get $16m^2 - 40m + 9 = 0$.
$\therefore$ slopes of tangents drawn from (4, 10) are $\dfrac{9}{4}$ & $\dfrac{1}{4}$.
Required tangent is $9x - 4y + 4 = 0$ or $x - 4y + 36 = 0$.

Q.10 (A)

Tangent to $y^2 - y + x = 0$ at (h, k) will be
$ky - \dfrac{y+k}{2} + \dfrac{x+h}{2} = 0$ or $(2k-1)y + x + h - k = 0$. {Using T = 0}
comparing this equation with $x + y - 1 = 0$ gives $h = 0$ & $k = 1$.

Q.11 (A)

Standard fact : Tangents at the extremities of any focal chord meet at a point on directrics.

Q.12 (A)

Standard result : $\left(at_1t_2, a(t_1 + t_2)\right)$.

Q.13 (D)

For tangent at $\left(\dfrac{at^2}{4}, \dfrac{at}{2}\right)$, slope = $\dfrac{1}{t}$.
Given that the tangent makes an angle of 45^o with x – axis hence slope = 1.
$\therefore$ t = 1 and the point of contact is $\left(\dfrac{a}{4}, \dfrac{a}{2}\right)$.

Q.14 (A)

Standard fact : Vertex is midpoint between directrics & focus.

Q.15 (D)

Let the common tangent be y = mx + c, then

for $y^2 = 4ax$, $c = \frac{a}{m}$ and

for x2 = 4by, $c = -bm^2$.

$\Rightarrow \frac{a}{m} = -bm^2$ or $m^3 = -\frac{a}{b}$.

Hence only one value of m is possible and thus only one common tangent may be drawn.

Q.16 (A)

Standard result : $t_2 = -t_1 - \frac{2}{t_1}$.

Q.17 (C)

Equation of normal with slope m to $y^2 = 12x$ is $y = mx - 6m - 3m^3$.
As slope of the given line is –1, hence equation of normal becomes $x + y = -9$.
$\therefore$ k = – 9.

Q.18 (C)

Equation of normal at $P(2t_1^2, 4t_1)$ to $y^2 = 8x$ is $t_1x + y = 4t_1 + 2t_1^3$.
As slope of the given line is 1, hence $t_1 = -1$.
Coordinates of P are (2,–4).

If this normal meets the parabola again at $Q(2t_2^2, 4t_2)$, then by $t_2 = -t_1 - \frac{2}{t_1}$, we get $t_2 = 3$.

Coordinates of Q are (18,12).
Now by distance formula, $PQ = 16\sqrt{2}$.

Q.19 (C)

Standard result : Any point on $y^2 = 4ax$ is of the form $(at^2, 2at)$.
$\{(a \sin^2 t, -2a \sin t)$ will represent only those points which lie between (a, –2a) to (a, 2a)$\}$.

Q.20 (D)

Any point on $y^2 = 8x$ is of the form $(2t^2, 4t)$ hence for P(2, 4), t = 1.

If normal at P meets the parabola again at $Q(2t_2^2, 4t_2)$, then by $t_2 = -t - \frac{2}{t}$, we get $t_2 = -3$.

Coordinates of Q are (18, –12).

Q.21 (B)

Diameter of $y^2 = 4ax$ corresponding to a chord of slope m is $y = \frac{2a}{m}$.

Hence diameter of $y^2 = x$ corresponding to the chord of slope 1 will be 2y = 1.

Q.22 (A)

Focus of $y^2 = ax$ is $\left(\frac{a}{4}, 0\right)$.

Substituting coordinates of focus in $2x - y - 8 = 0$ gives $a = 16$.
Now directrics will be $x + 4 = 0$.

Q.23 (D)

$S : y^2 - 16x$,
$S_1 : -15$,
$T : y - 8(x + 1)$.
By $T = S_1$, equation of chord with (1, 1) as midpoint will be
$y - 8(x + 1) = -15$ or $8x - y = 7$.

Q.24 (A)

Any tangent of slope m to $(x - 6)^2 + y^2 = 2$ will be
$y = m(x-6) \pm \sqrt{2}\sqrt{1+m^2}$.
Also focus of $y^2 = 16x$ is (4, 0).
As the tangent passes through the focus, hence by substituting coordinates of focus in the equation of the tangent we get $m^2 = 1$.

Q.25 (C)

Any tangent to $y^2 = 4x$ will be $y = mx + \frac{1}{m}$.

As it is drawn through (1, 4) hence $4 = m + \frac{1}{m}$ or $m^2 - 4m + 1 = 0$.

Now $m_1 + m_2 = 4$ & $m_1 m_2 = 1$ gives $m_1 - m_2 = 2\sqrt{3}$.
Now angle between the tangents will be given by
$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \Rightarrow \tan\theta = \sqrt{3}$ or $\theta = \frac{\pi}{3}$.

Q.26 (C)

Standard fact : If three normal can be drawn to $y^2 = 4ax$ from (h, 0), then $h > 2a$.

Q.27 (C)

Let coordinates of B be $(at^2, 2at)$, then slope of AB = $\frac{2}{t}$ and therefore slope of BC = $-\frac{t}{2}$.

Now equation of BC will be $y - 2at = -\frac{t}{2}\left(x - at^2\right)$.

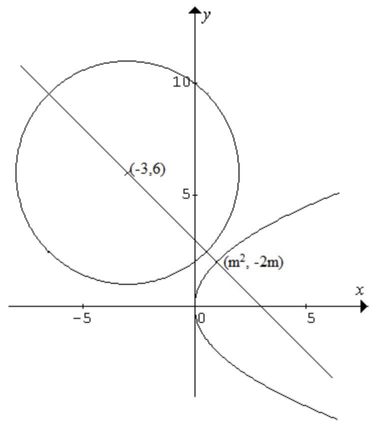
Coordinates of C lying on x – axis will be $\left(at^2 + 4a, 0\right)$.

Projection of B on x – axis will be $(at^2, 0)$.
Hence projection of BC on x – axis = 4a.

Q.28 (B)

Standard fact : Sum of ordinates of feet of any three concurrent normal to $y^2 = 4ax$ is Zero.

Q.29 (A)



For shortest distance the two curve must have a common normal.
Also in a circle, normal passes through the center.

Now any normal to $y^2 = 4x$ will be $y = mx - 2m - m^3$ and center of the circle is (–3, 6).

Substituting coordinates of center in equation of normal gives $m^3 + 5m + 6 = 0$ or $m = -1$.

Now $m = -1$ gives $t = 1$ and foot of normal as (1, 2).

Distance of (1, 2) from center of the circle is $4\sqrt{2}$ and radius of the circle is 5.

Shortest distance between the two curves is $4\sqrt{2} - 5$.

Q.30 (A)

Any normal to $y^2 = 4ax$ will be $y = mx - 2am - am^3$

and that to $y^2 = 4c(x - d)$ will be $y = m(x - d) - 2cm - cm^3$

for common normal, $2am + am^3 = dm + 2cm + cm^3$

or $m^2 = \dfrac{d + 2c - 2a}{a - c}$.

Hence $2c + d > 2a$, as it is given that $a > c$.

Q.31 (C)

The question is to find the Pole of $x - y + 3 = 0$ w.r. to $y^2 = 8x$.
Let the pole be (h, k), then the polar will be $ky = 4(x + h)$ or $4x - ky + 4h = 0$.
Comparing this equation with $x - y + 3 = 0$ gives $h = 3$ & $k = 4$.

Q. 32 (B)

As ordinate of C is zero and C divides AB in 1 : 2, hence

$\dfrac{4at_1 + 2at_2}{4} = 0 \Rightarrow 2t_1 + t_2 = 0.$

Q.33 (D)

If Ordinate = Abscissa, then the point is A(4a, 4a).
By comparing these coordinates with $(at^2, 2at)$ we get $t = 2$.

For other end point $B(at_1^2, 2at_1)$ of normal $t_1 = -t - \dfrac{2}{t}$ gives $t_1 = -3$.

Coordinates of B are (9a, – 6a), Also the focus is S(a, 0).

Now slope of SA = $\dfrac{4}{3}$ & slope of OB $= -\dfrac{3}{4}$.

Hence angle between OA & OB will be $\dfrac{\pi}{2}$.

Q.34 (C)

For shortest distance, the two curve must have a common normal.
Also in a circle, normal passes through the center.

Now any normal to $y^2 = 4x$ will be $y = mx - 2m - m^3$ and center of the circle is (0,12).

Substituting coordinates of center in equation of normal gives

$m^3 + 2m + 12 = 0$ or $m = -2$.

Now $m = -2$ gives $t = 2$ and foot of normal as (4, 4).

Q.35 (B)

Slope of normal at $P(at^2, 2at) = -t$

$\therefore\ t = -\tan\phi$.

Now if this normal meets the parabola again at $Q(at_1^2, 2at_1)$, then $t_1 = -t - \frac{2}{t}$.

Also slope of tangent to the parabola at Q = $\frac{1}{t_1}$.

Angle between tangent at Q & normal at P will be given by $\tan\theta = \dfrac{\frac{1}{t_1} + t}{1 - \frac{t}{t_1}}$ or $\tan\theta = \frac{t}{2}$.

$\Rightarrow \tan\theta = \frac{1}{2}\tan\phi$.

Q.36 (C)

Slope of focal chord having one end at $P(at^2, 2at)$, $t > 0$, is $\frac{2t}{t^2 - 1}$ and length is $l = a\left(t + \frac{1}{t}\right)^2$.

Now equation of focal chord will be $2tx - \left(1 - t^2\right)y = 2at$.

Distance of this line from the origin, $\dfrac{2at}{\sqrt{4t^2 + \left(1 - t^2\right)^2}} = p$ i.e. $\frac{2at}{1 + t^2} = p$.

Hence $l = \frac{4a^3}{p^2}$.

Q.37 (C)

Let the vertices of the triangle be P(0,0), $Q(at^2, 2at)$ & $R\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.

Now area of triangle, $A = \frac{1}{2}\left\|\begin{matrix} 1 & 0 & 0 \\ 1 & at^2 & \frac{a}{t^2} \\ 1 & 2at & -\frac{2a}{t} \end{matrix}\right\|$ or $A = a^2\left|t + \frac{1}{t}\right|$.

Hence absolute value of difference of ordinates of Q & R i.e. $2a\left|t + \frac{1}{t}\right|$ is $\frac{2A}{a}$.

Q.38 (C)

Let P & Q be $(at_1^2, 2at_1)$ & $(at_2^2, 2at_2)$ such that $t_2 = -t_1 - \frac{2}{t_1}$.

Also A is (0,0).

Now slope of AQ = $\frac{2}{t_2}$.

Equation PR will be $y - 2at_1 = \frac{2}{t_2}\left(x - at_1^2\right)$.

R lies on x – axis hence coordinates of R will be $\left(at_1^2 - at_1t_2, 0\right)$.

Length of AR = $\left|at_1^2 - at_1t_2\right|$ or $2a\left(t_1^2 + 1\right)$.

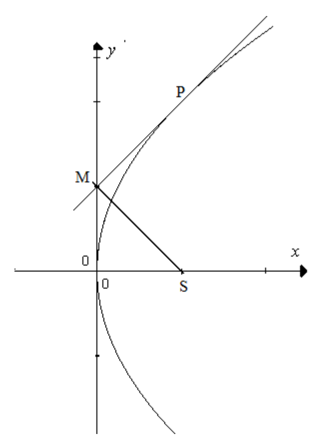
Hence AR = twice the focal distance of P.

Q.39 (B)

Standard fact : Normals at the extremities of a focal chord are mutually perpendicular.

Q.40 (A)

Let $\alpha = t^2$ & $\beta = 2t$, then eq. of tangent at P will be $ty = x + t^2$.
As foot of perpendicular from focus on any tangent lies on tangent at the vertex i.e. y – axis hence coordinates of M will be (0, t)
Now area of triangle formed by $P\left(t^2, 2t\right)$, $M(0, t)$ & $S(1, 0)$



$$A = \frac{1}{2}\begin{vmatrix} 1 & t^2 & 2t \\ 1 & 0 & t \\ 1 & 1 & 0 \end{vmatrix} = \frac{t\left(t^2 + 1\right)}{2}$$

As $0 \le \beta \le 2$ hence $0 \le t \le 1$. Maximum area will be = 1.