Solutions
PARABOLA
Ex. 3

Q.1

Let P be (h, k). Also let tangents from P be $t_1 y = x + at_1^2$ & $t_2 y = x + at_2^2$, where points of contact of these tangents being $Q\left(at_1^2, 2a_1\right) \& R\left(at_2^2, 2a_2\right)$.

Now point of intersection of tangents will be $h = at_1 t_2, k = a\left(t_1 + t_2\right)$. Area of triangle PQR will now be given by

$$\frac{1}{2}\left\|\begin{vmatrix} 1 & at_1^2 & 2at_1 \\ 1 & at_2^2 & 2at_2 \\ 1 & at_1 t_2 & a\left(t_1 + t_2\right) \end{vmatrix}\right\| = 4a^2 \text{ which implies } \left(t_1 - t_2\right)^2 = 4.$$

But $h = at_1 t_2, k = a\left(t_1 + t_2\right) \Rightarrow a^2\left(t_1 - t_2\right)^2 = k^2 - 4ah$, hence $k^2 - 4ah = 16a^2$.

Required locus is $y^2 = 4a\left(x - 4a\right)$ which is a parabola.

Q.2

Let P & Q be $\left(at_1^2, 2at_1\right) \& \left(at_2^2, 2at_2\right)$, then $t_2 = -t_1 - \dfrac{2}{t_1}$.

Now $OQ^2 = a\left(t_2^4 + 4t_2^2\right)$ or $OQ^2 = a^2\left(\left(t_1 + \dfrac{2}{t_1}\right)^4 + 4\left(t_1 + \dfrac{2}{t_1}\right)^2\right)$

$\Rightarrow OQ^2 = a^2\left(\left(t_1 + \dfrac{2}{t_1}\right)^2 + 2\right)^2 - 4a^2$. But by A.M. $\geq$ G.M., $\left|t_1 + \dfrac{2}{t_1}\right| \geq 2\sqrt{2}$.

$\Rightarrow \left(\left(t_1 + \dfrac{2}{t_1}\right)^2 + 2\right)^2 \geq 100$. Hence $\left|OQ\right| \geq 4a\sqrt{6}$.

Q.3

If normal at $P\left(t_1\right) \& \left(t_2\right)$ meet on the parabola, then $t_1 t_2 = 2$.

Also P,Q,R & N(point of intersection of normals) will form a cyclic quadrilateral and circle passing through P, Q & R will have RN as diameter as $\angle RPN = \dfrac{\pi}{2}$.

Now coordinates of R will be $\left(at_1 t_2, a\left(t_1 + t_2\right)\right)$ or $\left(2a, a\left(t_1 + t_2\right)\right)$. Similarly coordinates of N will be $\left(a\left(t_1^2 + t_2^2 + t_1 t_2 + 2\right), -at_1 t_2\left(t_1 + t_2\right)\right)$ or $\left(a\left(t_1^2 + t_2^2 + 4\right), -2a\left(t_1 + t_2\right)\right)$

Now let the circum center be (h, k), then

$$h = \frac{a\left(t_1^2 + t_2^2 + 6\right)}{2} \& k = -\frac{a\left(t_1 + t_2\right)}{2} \Rightarrow \frac{2h}{a} - 6 = t_1^2 + t_2^2 \& \frac{4k^2}{a^2} = t_1^2 + t_2^2 + 2t_1 t_2$$

Or eliminating t gives & replacing (h, k) with (x, y) gives required locus as $2y^2 = a\left(x - a\right)$.

Q.4

Substituting $y = ax^2 - b$ in $x^2 + y^2 = 1$ gives $x^2 + \left(ax^2 - b\right)^2 = 1$ or $a^2x^4 + \left(1 - 2ab\right)x^2 + b^2 - 1 = 0$.

Now for four distinct points of intersection the above equation must have four distinct real roots.
As the given equation is a biquadratic so considering $x^2 = t$ gives a quadratic in t both of whose roots must be real & positive.

Hence $a^2, 2ab - 1, b^2 - 1$ must be of same sign and $\left(1 - 2ab\right)^2 > 4a^2\left(b^2 - 1\right)$.

$\Rightarrow 2ab > 1, b > 1, 4a^2 - 4ab + 1 > 0$.

Clearly if a > b > 1, then all the above conditions get satisfied.
(remember here that a > b > 1 is a sufficient condition and may not be necessary)

Q.5

Let P, Q, P'& Q' be $\left(at_1^2, 2at_1\right), \left(at_2^2, 2at_2\right), \left(at_3^2, 2at_3\right) \& \left(at_4^2, 2at_4\right)$.

Now PQ is a focal chord & PP', QQ' are normal chords hence $t_2 = -\frac{1}{t_1}, t_3 = -t_1 - \frac{2}{t_1} \& t_4 = \frac{1}{t_1} + 2t_1$.

Slope of PQ = $\frac{2t_1 - 2t_2}{t_1^2 - t_2^2}$ or $\frac{2t_1}{t_1^2 - 1}$. Similarly

Slope of P'Q'= $\frac{2t_4 - 2t_3}{t_4^2 - t_3^2}$ or $\frac{2t_1}{t_1^2 - 1}$, hence PQ is parallel to P'Q'.

Also $PQ = a\left(t_1 + \frac{1}{t_1}\right)^2$ & P'Q' = $a\sqrt{\left(t_3^2 - t_4^2\right)^2 + 4\left(t_3 - t_4\right)^2}$ or $a\left|t_3 - t_4\right|\sqrt{\left(t_3 + t_4\right)^2 + 4}$

$\Rightarrow P'Q' = 3a\left(t_1 + \frac{1}{t_1}\right)^2$, hence P'Q' = 3 PQ.

Q.6

Let the fixed point on axis be P(h, 0), then any line passing through this point will be y = m(x – h).
Substituting $\left(at^2, 2at\right)$ this gives $amt^2 - 2at - hm = 0$.

$\Rightarrow t_1 + t_2 = \frac{2}{m} \& t_1t_2 = -\frac{h}{a}$, where t_1 & t_2 are parameters of those points where this line meets the parabola. Also $t_1^2 + t_2^2 = \frac{4}{m^2} + \frac{2h}{a}$.

Now circle having this chord as diameter will be

$x^2 + y^2 - a\left(t_1^2 + t_2^2\right)x - 2a\left(t_1 + t_2\right)y + a^2t_1^2t_2^2 + 4a^2t_1t_2 = 0$.

Or $x^2 + y^2 - a\left(\frac{4}{m^2} + \frac{2h}{a}\right)x - \frac{4a}{m}y + h^2 - 4ah = 0$.

Now if we consider two such circles with $m = m_1$ & $m = m_2$, then radical axis of these circles will be

$\left(\frac{1}{m_1^2} - \frac{1}{m_2^2}\right)x + \left(\frac{1}{m_1} - \frac{1}{m_2}\right)y = 0$ or $\left(m_1 + m_2\right)x + m_1m_2y = 0$.

Clearly it passes through the origin.

Q.7

Comparing P(16, 16) with $\left(4t^2, 8t\right)$ gives t = 2.
Now tangent at P will be 2y = x + 16 & normal at P will be 2x + y = 48.
Points where these lines meet the x-axis will be A(-16, 0) & B(24, 0).
As angle APB is a right angle hence the circle passing through P, A & B will have AB as diameter.
Hence $C_1 : \left(x+16\right)\left(x-24\right)+y^2=0$.
Equation of common chord of C_1 & C_2 will be $6x+y+197=0$.

Q.8

Let $l = at^2 \,\&\, m = 2at$. Now vertices of the triangle are $A\left(0,2\right), B\left(0,\frac{1}{2at}\right) \,\&\, C\left(\frac{1-4at}{at^2},2\right)$.

As the triangle is right angled hence by the concept of Euler's line its circum center (x, y) will be $\left(0+0+\frac{1-4at}{at^2}, 2+\frac{1}{2at}+2\right)$.

Now $t=\frac{1}{2a\left(y-4\right)} \Rightarrow x=4a\left(y-6\right)\left(y-4\right)$, which is equation of a parabola.

Q.9

Any tangent to $y^2=4ax$ will be $y=mx+\frac{a}{m}$ and any normal to $x^2=4by$ will be $y=mx+2b+\frac{b}{m^2}$

Comparing the two equations gives $\frac{a}{m}=2b+\frac{b}{m^2}$ or $2bm^2-am+b=0$.

For this equation to have real & distinct roots $a^2>8b^2$.

Q.10

Let B & C be $\left(at_1^{\,2},2at_1\right) \& \left(at_2^{\,2},2at_2\right)$ such that A is $\left(at_1t_2, a\left(t_1+t_2\right)\right)$. Also let another tangent be drawn at $D\left(at_3^{\,2},2at_3\right)$ such that $P \,\&\, Q$ are $\left(at_1t_3, a\left(t_1+t_3\right)\right) \& \left(at_2t_3, a\left(t_2+t_3\right)\right)$.

Now $AP=a\left|t_2-t_3\right|\sqrt{t_1^{\,2}+1}, AQ=a\left|t_3-t_1\right|\sqrt{t_2^{\,2}+1}$.

Also $AB=a\left|t_2-t_1\right|\sqrt{t_1^{\,2}+1} \,\&\, AC=a\left|t_2-t_1\right|\sqrt{t_2^{\,2}+1}$.

$\Rightarrow \frac{AP}{AB}+\frac{AQ}{AC}=\frac{\left|t_2-t_3\right|+\left|t_3-t_1\right|}{\left|t_2-t_1\right|}$.

Now considering t_1, t_3 & t_2 in cyclic order we get $\frac{AP}{AB}+\frac{AQ}{AC}$ = 1.

Q.11

Let the point K be (h,0) and slope of chord through K be tan θ, then any point on this line at a distance r from K will be (h + r cos θ, r sin θ).
For r = PK & r = QK, this point will satisfy the equation of parabola, hence by substituting these coordinates in the equation of the parabola we get $\left(\sin^2\theta\right)r^2-\left(4a\cos\theta\right)r-4ah=0$.

Roots of this equation are PK & -QK, hence PK – QK = $\frac{4a\cos\theta}{\sin^2\theta}$, & PK.QK = $\frac{4ah}{\sin^2\theta}$.

Now $\dfrac{1}{PK^2}+\dfrac{1}{QK^2}=\dfrac{(PK-QK)^2+2PK.QK}{(PK.QK)^2} \Rightarrow \dfrac{1}{PK^2}+\dfrac{1}{QK^2}=\dfrac{16a^2\cos^2\theta+8ah\sin^2\theta}{64a^2h^2}$

Clearly if h = 2a, then $\dfrac{1}{PK^2}+\dfrac{1}{QK^2}=\dfrac{1}{4h^2}$.

Q.12

Any tangent to $y^2=4a(x+a)$: $y=mx+am+\dfrac{a}{m}$

& an orthogonal tangent to $y^2=4b(x+b)$: $y=-\dfrac{1}{m}x-\dfrac{b}{m}-bm$.

Arranging both the equations as quadratic equations in m gives
$bm^2+my+x+b=0$ & $(x+a)m^2-ym+a=0$.

Comparing the two equations gives $\dfrac{b}{x+a}=\dfrac{y}{-y}=\dfrac{x+b}{a} \Rightarrow x+a+b=0$.

Now combining $y^2=4a(x+a)$ & $y^2=4b(x+b)$ in order get a linear equation we get common chord as x + a + b = 0.

Q.13

Let the fixed parabola be $y^2=4ax$ & the variable parabola be $(y-k)^2=-4a(x-h)$ having vertex at P(h, k).

Now as the two parabolas touch hence tangent to the two parabolas at $(at^2,2at)$ must be the same.

Tangent to $y^2=4ax$ will be $y=mx+\dfrac{a}{m}$ touching it at $T\left(\dfrac{a}{m^2},\dfrac{2a}{m}\right)$ &

that to $(y-k)^2=-4a(x-h)$ will be $y-k=m(x-h)-\dfrac{a}{m}$ or $y=mx+k-mh-\dfrac{a}{m}$.

Comparing the two equations gives $k-mh=\dfrac{2a}{m}$ & substituting coordinates of T in equation of

variable parabola gives $\left(\dfrac{2a}{m}-k\right)^2=-4a\left(\dfrac{a}{m^2}-h\right)$.

Or $hm^2-km+2a=0$ & $(k^2-4ah)m^2-4akm+8a^2=0$.

Comparing the two equations in order to eliminate m gives
$\dfrac{k^2-4ah}{h}=4a$ or $k^2=8ah$, hence required locus is $y^2=8ax$.

Q.14

Adding the two equations gives $x^2+6x-4y+13=0$ or $(x+3)^2=4(y-1)$, which means each of the points A, B, C & D lie on a parabola with vertex at (-3, 1) and focus at P(-3, 2).

Hence PA, PB, PC, PD will be focal distances of these points.

Now let any point on this parabola be $(2t-3,t^2+1)$. Substitute these coordinates in the equation
$x^2-y^2+6x+16y-46=0$ to get $t^4-18t^2+24t+40=0$.

Now let the roots of this be t_1,t_2,t_3,t_4, then

$t_1+t_2+t_3+t_4=0, t_1t_2+t_1t_3+t_1t_4+t_2t_3+t_2t_4+t_3t_4=-18$

Also Focal distance of a point with parameter t will be $1+t^2$, hence

PA + PB + PC + PD = 4 + $t_1^2+t_2^2+t_3^2+t_4^2$.

Now from above relations

$t_1^2+t_2^2+t_3^2+t_4^2=(t_1+t_2+t_3+t_4)^2-2(t_1t_2+t_1t_3+t_1t_4+t_2t_3+t_2t_4+t_3t_4)$

Therefor PA + PB + PC + PD = 40.

Q.15

Normal to $y^2=4ax$ at any point P(t) will be $tx+y=2at+at^3$.

This will meet the x-axis at $Q(2a+at^2,0)$.

The line perpendicular to normal and passing through Q will be $x-ty=2a+at^2$.

Now this equation may be rearranged as $y=m(x-2a)-\frac{a}{m}$, where $m=\frac{1}{t}$.

Clearly its in form of tangent line of slope m to the parabola $y^2=-4a(x-2a)$.

Q.16

Let mid point of any such chord be $M(at^2,2at)$.

Now using T = S_1, equation of chord of $x^2+y^2=16a^2$ having mid point at M may be represented as $at^2x+2aty=a^2t^4+4a^2t^2$.

As this chord is drawn through (h, 0) hence substituting these coordinates in equation of chord we get $at^2x+2aty=a^2t^4+4a^2t^2$.

Now the above equation gives three values of t, namely $0 \,\&\, \pm\sqrt{\frac{h-4a}{a^2}}$ out of which the later two values will be real & other than 0 only if h > 4a.

Also for M to be mid point of chord it must lie inside the circle hence

$a^2t^4+4a^2t^2-16a^2<0$ or $\left(\frac{4a-h}{a^2}\right)^2+4\left(\frac{4a-h}{a^2}\right)-16<0$, hence $h<(\sqrt{5}+1)2a$.

Q.17

Let A, B & P be $(at_1^2,2at_2),(at_2^2,2at_2)\&(at_1t_2,a(t_1+t_2))$ on $y^2=4ax$.

Tangent PB will be $x-t_2y+at_2^2=0$.

Now any circle touching PB at P may be represented as family of point circle having center at P and the line PB i.e. $(x-at_1t_2)^2+(y-a(t_1+t_2))^2+l\,(x-t_2y+at_2^2)=0$.

As this circle passes through F(a, 0), hence $l=-a(1+t_1^2)$.

Now the circle touching PB at P & passing through F is

$(x-at_1t_2)^2+(y-a(t_1+t_2))^2-a(1+t_1^2)(x-t_2y+at_2^2)=0$

Substituting coordinates of A in L.H.S. of equation of circle gives

$(at_1^2-at_1t_2)^2+(2at_1-a(t_1+t_2))^2-a(1+t_1^2)(at_1^2-2at_1t_2+at_2^2)$ or $(t_1-t_2)^2(t_1^2+1-(1+t_1^2))$ which is zero, hence this circle passes through A.

Q.18

(i) Let P, Q & R be the vertices of a triangle formed by three tangents of $y^2 = 4ax$, then the coordinates of these points can be taken as $\left(at_1t_2, a\left(t_1+t_2\right)\right), \left(at_2t_3, a\left(t_2+t_3\right)\right) \& \left(at_3t_1, a\left(t_3+t_1\right)\right)$. Also the focus is S(a, 0).

Now $m_{PQ} = \frac{1}{t_2}, m_{PR} = \frac{1}{t_1}, m_{FQ} = \frac{t_2+t_3}{t_2t_3-1} \;\&\; m_{FR} = \frac{t_3+t_1}{t_3t_1-1}$.

Let angle between PQ & PR be α, then $\tan\alpha = \dfrac{\frac{1}{t_2}-\frac{1}{t_1}}{1+\frac{1}{t_2}\frac{1}{t_1}}$ i.e. $\frac{t_1-t_2}{t_1t_2+1}$.

Similarly let angle between FQ & FR be β, then $\tan\beta = \dfrac{\frac{t_3+t_1}{t_3t_1-1}-\frac{t_2+t_3}{t_2t_3-1}}{1+\frac{t_3+t_1}{t_3t_1-1}\frac{t_2+t_3}{t_2t_3-1}}$ i.e. $\frac{t_2-t_1}{t_1t_2+1}$.

(Here take care to put slopes in same cyclic order to get correct angles)

Clearly $\alpha \,\&\, \beta$ are supplementary angles, hence PQFR is a cyclic quadrilateral.

(ii) Altitude through P must be perpendicular to tangent QR, hence its slope will be $-t_3$.

Equation of this altitude will be $y - a\left(t_1+t_2\right) = -t_3\left(x - at_1t_2\right)$.

Similarly altitude through Q will be $y - a\left(t_2+t_3\right) = -t_1\left(x - at_2t_3\right)$.

Eliminating y between these two equations gives $x = -a$, hence orthocenter lies on directrics.

Q.19

Any circle touching the parabola at P(t) will also touch the tangent to parabola at P.

Now any circle touching the line $y = tx + at^2$ at P may be represented as family of point circle having center at P and the line PB i.e. $\left(x - at^2\right)^2 + \left(y - 2at\right)^2 + \lambda\left(x - ty + at^2\right) = 0$.

As this circle passes through F(a, 0), hence $\lambda = -a\left(1+t^2\right)$.

Hence the circle touching the parabola at P & passing through F is

$$\left(x - at^2\right)^2 + \left(y - 2at\right)^2 - a\left(1+t^2\right)\left(x - ty + at^2\right) = 0$$

Similarly the circle touching the parabola at Q & passing through F is

$\left(x - \frac{a}{t^2}\right)^2 + \left(y + \frac{2a}{t}\right)^2 - a\left(1+\frac{1}{t^2}\right)\left(x + \frac{1}{t}y + \frac{a}{t^2}\right) = 0$, note that P & Q are end points of a focal chord.

The two equation of circles simplify to

$$x^2 + y^2 - a\left(3t^2+1\right)x + at\left(t^2-3\right)y + 3a^2t^2 = 0 \;\&\; x^2 + y^2 - a\left(\frac{3}{t^2}+1\right)x + \frac{a}{t}\left(3 - \frac{1}{t^2}\right)y + \frac{3a^2}{t^2} = 0.$$

Now $g_1 = \frac{a\left(3t^2+1\right)}{2}, f_1 = -\frac{at\left(t^2-3\right)}{2}, c_1 = 3a^2t^2 \;\&\; g_2 = \frac{a\left(3+t^2\right)}{2t^2}, f_2 = -\frac{a\left(3t^2-1\right)}{2t^3}, c_2 = \frac{3a^2}{t^2}$ gives

$$2g_1g_2 + 2f_1f_2 = 2\frac{a\left(3t^2+1\right)}{2}\frac{a\left(3+t^2\right)}{2t^2} + 2\frac{at\left(t^2-3\right)}{2}\frac{a\left(3t^2-1\right)}{2t^3} \text{ or } 2g_1g_2 + 2f_1f_2 = a^2\left[\frac{3t^4+3}{t^2}\right] = c_1 + c_2.$$

Hence the circles are orthogonal.

Q.20

Equation of line joining (1, 0) & (0, 2) is 2x + y = 2.

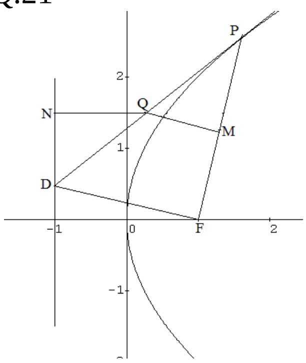
Now any curve having xy = 0 as pair of tangents and 2x + y – 2 = 0 as chord of contact may be represented as $(2x+y-2)^2+\lambda xy=0$ or $4x^2+(\lambda+4)xy+y^2-8x-4y+4=0$.

For this equation to represent a parabola $h^2=ab\Rightarrow\left(\frac{\lambda+4}{2}\right)^2=4\Rightarrow\lambda=-8$ or 0.

But for $\lambda=0$ the equation becomes $(2x+y-2)^2=0$.

Hence required parabola is $4x^2-4xy+y^2-8x-4y+4=0$.

Q.21



Consider the parabola $y^2=4ax$.

Let P be $(at^2, 2at)$ & Q be (h, k). Also equation of PQ will be

$ty=x+at^2$, hence $k=\frac{h+at^2}{t}$.

Now slope of FP is $\frac{2t}{t^2-1}$, hence equation of QM will be

$(y-k)=\frac{1-t^2}{2t}(x-h)$.

Also QN will be parallel to x-axis thus its equation will be y = k.

Hence QN = a + h.

Now perpendicular distance of QM from F i.e. FM = $\left|\frac{k+\frac{1-t^2}{2t}(a-h)}{\sqrt{\left(\frac{1-t^2}{2t}\right)^2+1}}\right|$.

Therefor $FM=\left|\frac{\frac{h+at^2}{t}+\frac{1-t^2}{2t}(a-h)}{\sqrt{\left(\frac{1-t^2}{2t}\right)^2+1}}\right|\Rightarrow FM=\left|\frac{(h+a)(1+t^2)}{\sqrt{(1-t^2)^2+4t^2}}\right|$

Or FM = h + a = QN.

Q.22

We can get the solution by first consider a fixed parabola touching the coordinate axes and then rotating it by an angle θ.

One such parabola is $x^2-2xy+y^2-2ax-2ay+a^2=0$ which touches the coordinate axes at (the end points of latus rectum i.e. (a, 0) & (0, a). Equation of its latus rectum is x + y = a.

Rotating the parabola by an angle θ transforms the equation of latus rectum into

$x(\cos\theta+\sin\theta)+y(\cos\theta-\sin\theta)=a$, which may be rearranged as

$y = -\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}x + \dfrac{a}{\cos\theta - \sin\theta}$. Now let $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = -m$, then the equation reduces to $y = mx - \dfrac{a}{\sqrt{2}}\sqrt{1+m^2}$, which is equation of tangent to the circle $x^2 + y^2 = \dfrac{a^2}{2}$.

Q.23

Let vertices of the triangle be $\left(at_1^2, 2at_1\right), \left(at_2^2, 2at_2\right), \left(at_3^2, 2at_3\right)$.

Now sides joining these will be

$$y = \frac{2}{t_1 + t_2}x + \frac{2at_1t_2}{t_1 + t_2}, y = \frac{2}{t_2 + t_3}x + \frac{2at_2t_3}{t_2 + t_3}, y = \frac{2}{t_3 + t_1}x + \frac{2at_2t_3}{t_3 + t_1}.$$

Let the first line touch $x^2 = 4by$, then $\dfrac{2at_1t_2}{t_1 + t_2} = -b\left(\dfrac{2}{t_1 + t_2}\right)^2$ or $at_1t_2\left(t_1 + t_2\right) = -2b$.

Similarly if the second line is a tangent then $at_2t_3\left(t_2 + t_3\right) = -2b$.

Now from these two conditions we get $t_1 + t_2 + t_3 = 0 \,\&\, t_2t_3t_1 = \dfrac{2b}{a}$.

Further $at_3t_1\left(t_3 + t_1\right) = a \times \dfrac{2b}{at_2} \times \left(-t_2\right) = -2b$, hence the third line also touch $x^2 = 4by$.

Q.24

If the triangle is equilateral, then its centroid will be same as circum center.

Let the vertices be $\left(at_1^2, 2at_1\right), \left(at_2^2, 2at_2\right), \left(at_3^2, 2at_3\right)$.

Centroid will be $h = \dfrac{a\left(t_1^2 + t_2^2 + t_3^2\right)}{3} \,\&\, k = \dfrac{2a\left(t_1 + t_2 + t_3\right)}{3}$.

Now consider the circle $x^2 + y^2 - 2hx - 2ky + c = 0$ and put $\left(at^2, 2at\right)$ in this equation to get $a^2t^4 + 2a\left(2a - h\right)t^2 - 4akt + c = 0$.

Now if this is the circum circle of triangle PQR, then t_1, t_2, t_3 will be three of its roots.

Using relations in roots and coefficients we get

$$t_1 + t_2 + t_3 + t_4 = 0, t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = \frac{2\left(2a - h\right)}{a} \,\&$$

$$t_1t_2t_3 + t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2 = \frac{4k}{a}.$$

Also $h = \dfrac{a\left(t_1^2 + t_2^2 + t_3^2\right)}{3} \,\&\, k = \dfrac{2a\left(t_1 + t_2 + t_3\right)}{3}$ gives $t_4 = -\dfrac{3k}{2a}$ & $t_1t_2 + t_1t_3 + t_1t_4 = \dfrac{12ah - 9k^2}{8a^2}$

Now from $t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = \dfrac{2\left(2a - h\right)}{a}$ we get

$$t_1t_2 + t_2t_3 + t_3t_1 + \left(t_1 + t_2 + t_3\right)t_4 = \frac{2\left(2a - h\right)}{a}.$$

Substituting the values gives $4ah - 9k^2 - 32a^2 = 0$.

Hence locus of centroid of triangle PQR is $9y^2 - 4ax + 32a^2 = 0$.

Q.25

Let extremities of the focal chord be $P\left(at^2,2at\right),Q\left(\frac{a}{t^2},-\frac{2a}{t}\right)$.

Point of intersection of tangents at P & Q will be $R\left(-a,2a\left(t-\frac{1}{t}\right)\right)$.

Now area of triangle PQR will be

$$\frac{1}{2}\left\|\begin{matrix}1 & at^2 & 2at\\ 1 & \frac{a}{t^2} & -\frac{2a}{t}\\ 1 & -a & a\left(t-\frac{1}{t}\right)\end{matrix}\right\| \text{ i.e. } \frac{a^2}{2}\left(t^2+\frac{1}{t^2}+2\right)\left|t+\frac{1}{t}\right|.$$

Similarly area of triangle OPQ will be

$$\frac{1}{2}\left\|\begin{matrix}1 & at^2 & 2at\\ 1 & \frac{a}{t^2} & -\frac{2a}{t}\\ 1 & 0 & 0\end{matrix}\right\| \text{ i.e. } a^2\left|t+\frac{1}{t}\right|.$$

Now ratio of these two area will be $\frac{1}{2}\left(t^2+\frac{1}{t^2}\right)+1$.

Q.26

Let slope of the variable line be $\tan\theta$.

Now any point on this line at a distance r from P(a, b) will be $\left(a+r\cos\theta,b+r\sin\theta\right)$.

These coordinates will satisfy $y^2=4cx$ for r = PA & r = PB.

Hence $\left(b+r\sin\theta\right)^2=4c\left(a+r\cos\theta\right)$ i.e. $\left(\sin^2\theta\right)r^2+\left(2b\sin\theta-4c\cos\theta\right)r+b^2-4ac=0$ will have PA & PB as roots. Now

$$PA+PB=\frac{4c\cos\theta-2b\sin\theta}{\sin^2\theta} \;\&\; PA\times PB=\frac{4ac-b^2}{\sin^2\theta}.$$

As given PA, PQ, PB are in H.P., hence $PQ=\frac{2\times PA\times PB}{PA+PB}\Rightarrow PQ=\frac{2\left(4ac-b^2\right)}{4c\cos\theta-2b\sin\theta}$.

Now let coordinate of Q be (x, y), then

$$x=a+PQ\cos\theta \;\&\; y=b+PQ\sin\theta\Rightarrow \cos\theta=\frac{x-a}{PQ} \;\&\; \sin\theta=\frac{y-b}{PQ}.$$

Substituting these in the expression of PQ we get $PQ=\frac{2\left(4ac-b^2\right)}{4c\left(x-a\right)-2b\left(y-b\right)}PQ$

or $2cx-by=6ac-2b^2$.

Hence locus of Q is a fixed straight line.

Q.27

Foot of perpendicular from the focus F on tangent at P will lie on y-axis, hence let P, F & M be $\left(at^2,2at\right),\left(a,0\right)\&\left(0,at\right)$.

Now area of triangle PFM will be

$$\frac{1}{2}\left\|\begin{matrix} 1 & at^2 & 2at \\ 1 & a & 0 \\ 1 & 0 & at \end{matrix}\right\| = \frac{a^2}{2}\left(t^3 + t\right).$$

Now range of t is 0 to 1.

Maximum area will be for t = 1 i.e. maximum area = a^2.

Q.28

Let mid point of any such chord be $M\left(at^2, 2at\right)$.

Now using T = S_1, equation of chord of $x^2 + y^2 = 16a^2$ having mid point at M may be represented as $at^2x + 2aty = a^2t^4 + 4a^2t^2$.

As this chord is drawn through (h, 0) hence substituting these coordinates in equation of chord we get $at^2x + 2aty = a^2t^4 + 4a^2t^2$.

Now the above equation gives three values of t, namely $0 \,\&\, \pm\sqrt{\frac{h - 4a}{a^2}}$ out of which the later two values will be real & other than 0 only if h > 4a.

Also for M to be mid point of chord it must lie inside the circle hence

$a^2t^4 + 4a^2t^2 - 16a^2 < 0$ or $\left(\frac{4a - h}{a^2}\right)^2 + 4\left(\frac{4a - h}{a^2}\right) - 16 < 0$, hence $h < \left(\sqrt{5} + 1\right)2a$.

Q.29

Reflection at a point P on any curved surface take place such that incident ray and reflected are reflections of each other in the normal to the curve at P.

Now y = b meets $y^2 = 4ax$ at point $P\left(\frac{b^2}{4a}, b\right)$. Comparing this with $\left(at^2, 2at\right)$ gives $t = \frac{b}{2a}$.

Normal to the parabola at this point will be

$\frac{b}{2a}x + y = 2a\frac{b}{2a} + a\left(\frac{b}{2a}\right)^3$ or $4abx + 8a^2y = 8a^2b + b^3$.

Now slope of normal is $-\frac{b}{2a}$ and y = b is parallel to x-axis so if θ is the angle between the incident ray and normal, then $\tan\theta = -\frac{b}{2a}$.

Reflected ray will make an angle 2θ with y = b, hence slope of reflected ray will be

$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{4ab}{4a^2 - b^2}$.

Equation of the reflected ray : $y - b = \frac{4ab}{4a^2 - b^2}\left(x - \frac{b^2}{4a}\right)$ or $4abx + \left(4a^2 - b^2\right)y = 4a^2b$.

Clearly (a, 0) satisfies this equation.

Q.30

If the triangle is equilateral, then its centroid will be same as circum center.

Let the vertices be $\left(at_1^2, 2at_1\right), \left(at_2^2, 2at_2\right), \left(at_3^2, 2at_3\right)$.

Centroid will be $h = \frac{a\left(t_1^2 + t_2^2 + t_3^2\right)}{3} \,\&\, k = \frac{2a\left(t_1 + t_2 + t_3\right)}{3}$.

Now consider the circle $x^2+y^2-2hx-2ky+c=0$ and put $\left(at^2,2at\right)$ in this equation to get $a^2t^4+2a\left(2a-h\right)t^2-4akt+c=0$.

Now if this is the circum circle of triangle PQR, then t_1,t_2,t_3 will be three of its roots.

Using relations in roots and coefficients we get

$$t_1+t_2+t_3+t_4=0,t_1t_2+t_1t_3+t_1t_4+t_2t_3+t_2t_4+t_3t_4=\frac{2\left(2a-h\right)}{a}\ \&$$

$$t_1t_2t_3+t_2t_3t_4+t_3t_4t_1+t_4t_1t_2=\frac{4k}{a}.$$

Also $h=\dfrac{a\left(t_1^2+t_2^2+t_3^2\right)}{3}\ \&\ k=\dfrac{2a\left(t_1+t_2+t_3\right)}{3}$ gives $t_4=-\dfrac{3k}{2a}$ & $t_1t_2+t_1t_3+t_1t_4=\dfrac{12ah-9k^2}{8a^2}$

Now from $t_1t_2+t_1t_3+t_1t_4+t_2t_3+t_2t_4+t_3t_4=\dfrac{2\left(2a-h\right)}{a}$ we get

$$t_1t_2+t_2t_3+t_3t_1+\left(t_1+t_2+t_3\right)t_4=\frac{2\left(2a-h\right)}{a}.$$

Substituting the values gives $4ah-9k^2-32a^2=0$.

Hence locus of centroid of triangle PQR is $9y^2-4ax+32a^2=0$.

Q.31

Given data implies that point of intersection of two normal lies on the parabola.

Let a normal be drawn at P(λ), then its equation will be $\lambda x+y=2a\lambda+a\lambda^3$.

If it passes through $\left(at^2,2at\right)$, then $\lambda at^2+2at=2a\lambda+a\lambda^3$.

$\Rightarrow a\lambda\left(t^2-\lambda^2\right)+2a\left(t-\lambda\right)=0$ or $\lambda^2+\lambda t+a=0$.

Q.32

Let the points on parabola be $A\left(at_1^2,2at_1\right),B\left(at_2^2,2at_2\right)\&C\left(at_3^2,2at_3\right)$.

Points of intersection of tangents at these points will be

$P\left(at_1t_2,a\left(t_1+t_2\right)\right),Q\left(at_2t_3,a\left(t_2+t_3\right)\right)\&R\left(at_3t_1,a\left(t_3+t_1\right)\right)$.

Now Area of $\Delta ABC=\dfrac{1}{2}\left\|\begin{matrix}1 & at_1^2 & 2at_1\\ 1 & at_2^2 & 2at_2\\ 1 & at_3^2 & 2at_3\end{matrix}\right\|$ & Area of $\Delta PQR=\dfrac{1}{2}\left\|\begin{matrix}1 & at_2t_3 & a\left(t_2+t_3\right)\\ 1 & at_3t_1 & a\left(t_3+t_1\right)\\ 1 & at_1t_2 & a\left(t_1+t_2\right)\end{matrix}\right\|$.

Now take the second determinant,

(i) Subtract $a\left(t_1+t_2+t_3\right)$ from third column to get

$$\text{Area of } \Delta PQR=\frac{1}{2}\left\|\begin{matrix}1 & at_2t_3 & -at_1\\ 1 & at_3t_1 & -at_2\\ 1 & at_1t_2 & -at_3\end{matrix}\right\|.$$

(ii) Multiply first row by t_1, second by t_2 & third by t_3 and take $t_1t_2t_3$ common from second column. Also take a common from second column and multiply 2a to first column to get

$$\text{Area of } \Delta PQR = \frac{1}{4}\left\|\begin{matrix} 2at_1 & 1 & at_1^{\,2} \\ 2at_2 & 1 & at_2^{\,2} \\ 2at_3 & 1 & at_3^{\,3} \end{matrix}\right\|.$$

(iii) Now interchange first column with second and then second with third to get

$$\text{Area of } \Delta PQR = \frac{1}{4}\left\|\begin{matrix} 1 & at_1^{\,2} & 2at_1 \\ 1 & at_2^{\,2} & 2at_2 \\ 1 & at_3^{\,3} & 2at_3 \end{matrix}\right\| = \frac{1}{2}\text{Area of } \Delta ABC.$$

Q.33

Let the points be $P\left(at_1^{\,2}, 2at_1\right) \& Q\left(at_2^{\,2}, 2at_2\right)$, where as given $t_2 = 2t_1$.

Now point of intersection of normal at P & Q will be

$x = a\left(t_1^{\,2} + t_2^{\,2} + t_1t_2 + 2\right) \& \; y = -at_1t_2\left(t_1 + t_2\right)$

Now $t_2 = 2t_1 \Rightarrow x - 2a = 7at_1^{\,2} \; \& \; y = -6at_1^{\,3}$.

Eliminating t_1 gives $36\left(x - 2a\right)^3 = 243ay^2$.

Q.34

Let the points be $P\left(at_1^{\,2}, 2at_1\right) \& Q\left(at_2^{\,2}, 2at_2\right)$, where as given $t_1t_2 = -1$.

Now point of intersection of normal at P & Q will be

$x = a\left(t_1^{\,2} + t_2^{\,2} + t_1t_2 + 2\right) \& \; y = -at_1t_2\left(t_1 + t_2\right)$.

Now $t_1t_2 = -1$ gives $x = a\left(t_1^{\,2} + t_2^{\,2} + 1\right) \& \; y = a\left(t_1 + t_2\right)$.

Eliminating t_1 & t_2 gives $a\left(x - 3a\right) = y^2$.

Q.35

Let the points be $P\left(at_1^{\,2}, 2at_1\right) \& Q\left(at_2^{\,2}, 2at_2\right)$, where as given $t_1t_2 = 2$.

Now mid point of P & Q will be $x = \dfrac{a\left(t_1^{\,2} + t_2^{\,2}\right)}{2} \& \; y = a\left(t_1 + t_2\right)$.

Now $t_1t_2 = 2$ gives $2x + 4a = a\left(t_1 + t_2\right)^2 \& \; y = a\left(t_1 + t_2\right)$.

Eliminating t_1 & t_2 gives $2a\left(x + 2a\right) = y^2$.