Solutions
PARABOLA
Ex. 2(C)

Q.1

Comparing $(2,4)$ with $(2t^2,4t)$ gives $t=1$.

Now other end of normal will be given by $t'=-t-\frac{2}{t}$ i.e. $t'=-3$.

Hence other end of normal is $(18,-12)$.

Now length of normal $=\sqrt{(2-18)^2+(4+12)^2}=16\sqrt{2}$. Thus $k=4$.

Q.2

Let mid point of PQ be (h,k), then equation of chord will be $ky-2ax=k^2-2ah$ $\{T=S_1\}$.

Also any tangent to $y^2=-4ax$ will be given by $ty+x=at^2$.

Comparing the two equations gives $\frac{t}{k}=\frac{1}{-2a}=\frac{at^2}{k^2-2ah}$ or $t=-\frac{k}{2a}$ & $at^2=\frac{2ah-k^2}{2a}$.

Eliminating t in these two relations gives $3k^2=4ah$.

Hence required locus is $y^2=\frac{4ax}{3}$.

Q.3

Shortest distance will be measured along common normal to the two curves.

Now any normal to $y^2=4x$ wil be $tx+y=2t+t^3$.

As this line is a normal to the given circle as well hence center $C(0,12)$ must satisfy it.

$\therefore t^3+2t-12=0$ or $(t-2)(t^2+2t+6)=0$.

Hence the normal will be drwan at $P(t=2)$ i.e. $P(4,4)$ to parabola.

Now distance of $P(4,4)$ from $C(0,12)$ is $4\sqrt{5}$.

Minimum distance will be $(CP-\text{radius})$ i.e. $4(\sqrt{5}-1)$. Thus $a+c-b=0$.

Q.4

Any tangent to $y^2=4ax$ will be $y=mx+\frac{a}{m}$. If this tangent is drwan from (h,k), then

$hm^2-km+a=0$. Let the roots be m_1 & m_2, then $m_1+m_2=\frac{k}{h}$ & $m_1m_2=\frac{a}{h}$.

Also given $m_1=\tan\alpha_1$ & $m_2=\tan\alpha_2$, hence $\frac{\cot\alpha_1}{\cot\alpha_2}=2\Rightarrow m_2=2m_1.\Rightarrow m_2=\frac{k}{3h}$ & $m_2{}^2=\frac{a}{2h}$.

Eliminating m_2 gives $2k^2=9ah$, hence required locus is $2y^2=9ax$.

Q.5

Any point on $y^2 = 4x$ will be $(t^2, 2t)$. Let the distance of this point from $(2,0)$ be d,

then $d^2 = (2-t^2)^2 + 4t^2$ or $d^2 = t^4 + 4$.

Hence minimum distance must be $d = 2$ and it will measured from $(0,0)$.

$a + b + 2 = 2$.

Q.6

Any normal to $y^2 = -8x$ will be $y = tx + 4t + 2t^3$.

Comparing this with $y = -2x - k$ gives $t = -2$ & $k = -4t - 2t^3$ i.e. $\frac{k}{3} = 8$.

Q.7

Any tangent to $y^2 = 16x$ will be $y = mx + \frac{4}{m}$ & that to $x^2 = 2y$ be $y = mx - \frac{m^2}{2}$.

Comparing the two tangents for being common gives $\frac{4}{m} = -\frac{m^2}{2}$ or $m = -2$.

Q.8

Let the feet of normals be $(2t_1^2, 4t_1), (2t_2^2, 4t_2)$ & $(2t_3^2, 4t_3)$.

Now for conormal points $t_1 + t_2 + t_2 = 0$, hence A.M. of ordinates i.e. $\frac{4t_1 + 4t_2 + 4t_2}{3} = 0$.

Q.9

For end points of focal chord $t_1 t_2 = -1$. Now for $A(4a, 4a), t_1 = 2$ hence $t_2 = -\frac{1}{2}$.

Similarly for other end $C(t_3)$ of normal at $B(t_2)$ we have $t_3 = -t_2 - \frac{2}{t_2}$ or $t_3 = \frac{9}{2}$

Also if four points $A(t_1), B(t_2), C(t_3)$ & $D(t_3)$ of a parabola are concyclic, then

$t_1 + t_2 + t_3 + t_4 = 0 \Rightarrow t_4 = -2 + \frac{1}{2} - \frac{9}{2} = -6$.

So coordinates of D are $p = 36a, q = -12a$ which gives $\frac{p+q}{4a} = 6$.

Q.10

The parabolas intersect at $(0,0)$ & $(4,4)$. Clearly the tangents at $(0,0)$ are the coordinate axes hence the parabola intersect at a right angle.

Now tangent to $y^2 = 4x$ at $(4,4)$ is $4y = 2(x+4)$ & that to $x^2 = 4y$ is $4x = 2(y+4)$. $\{\text{By } T = 0\}$

Slopes of these tangents are $m_1 = \frac{1}{2}$ & $m_2 = 2$.

Now $\tan\theta = \dfrac{2-\frac{1}{2}}{1+2\times\frac{1}{2}} \Rightarrow \theta = \tan^{-1}\left(\frac{3}{4}\right)$, hence $y - x = 1$.

Q.11

Let mid point of chord be (p,q), then equation of chord will be

$7qy - 25\frac{x+p}{2} = 7q^2 - 25p$ $\{\text{By } T = S_1\}$

Slope $= \frac{25}{14q} = 4 \Rightarrow 56q = 25$. Hence required slope is $56y = 25$.

Q.12

Equation of tangent to $y^2 = 8x$ will be $x - ty + 2t^2 = 0$.

If it is a tangent to $x^2 + y^2 + 6x = 0$, then distance of this line from center of the circle i.e. $(-3,0)$ will be equal to radius i.e. 3.

$\therefore \left|\dfrac{-3-0+2t^2}{\sqrt{1+t^2}}\right| = 3$ or $4t^4 - 21t^2 = 0$.

Hence 3 values of t are possible which means there will be three common tangents.

Q.13

For $y^2 = 4ax$ number of normals from (h,k) will be

3 if $27ay^2 - 4(x-2a)^2 < 0$

2 if $27ay^2 - 4(x-2a)^2 = 0$

1 if $27ay^2 - 4(x-2a)^2 > 0$

Hence for $y^2 = 6x$ & $(2,8)$, $27ay^2 - 4(x-2a)^2 = 27\times\frac{3}{2}\times 64 - 4\times(2-3)^3 > 0$.

Only one normal can be drawn.

Q.14

Let focus be S, point of intersection of tangent at P with axis be T & point of intersection of normal at P with axis be N, then by properties of parabola SP=ST=SN.

Hence SP $=\frac{TN}{2}$ i.e. $\frac{\sqrt{29}}{2}$. Now $\frac{\sqrt{14k+1}}{k}=\frac{\sqrt{29}}{2}\Rightarrow k=2.$

Q.15

Any point on $y^2=-8(x+4)$ will be $(-2t^2-4,4t)$.

Chord of contact of this point w.r.to $y^2=4x$ will be $4ty=2x-2(2t^2+4)$. $\{\text{By } T=0\}$

Now if mid point of this chord is (h,k), then by $T=S_1$ eq. of this chord will be $ky=2x+k^2-2h$.

Comparing the two eqs. gives $t=\frac{k}{4}$ & $2(2t^2+4)=-k^2+2h$.

Eliminating t gives $2\left(\frac{k^2}{8}+4\right)=-k^2+2h$.

Required locus is $5y^2=8(x-4)$ whose Latus rectum is $\frac{8}{5}$.

Q.16

Tangent to $y^2=4x:y=mx+\frac{1}{m}$ & Tangent to $x^2=-32y:y=mx+8m^2$.

Comparing the two eqs. gives $\frac{1}{m}=8m^2$ or $m=\frac{1}{2}$.

Hence common tangent is $y=\frac{x}{2}+2$ or $x-2y+4=0$.

Q.17

By properties of a parabola $\frac{SP+SQ}{SP\times SQ}=\frac{1}{a}\Rightarrow a=\frac{1}{7}$.

Now $L=2a\Rightarrow 21L=6$.

Q.18

Both the parabolas are symmatric about the line x = y, hence there common normal must be perpendicular to x = y i.e. of slope -1.

Equation of normal to $y^2=4\times\frac{1}{4}(x-1)$ with slope m is $y=m(x-1)-\frac{1}{2}m-\frac{1}{4}m^3$, which is normal at $\left(\frac{m^2}{4}+1,-\frac{m}{2}\right)$ i.e. $P\left(\frac{5}{4},\frac{1}{2}\right)$.

Clearly for the other parabola the point at which common normal is drwan will be $Q\left(\frac{1}{2},\frac{5}{4}\right)$

$\left(\text{image of } P \text{ in } x = y\right)$.

$PQ = \frac{3\sqrt{2}}{4} \Rightarrow k = 4.$

Q.19

Center of $(x-6)^2 + y^2 = 2$ is $(6,0)$ & $r = \sqrt{2}$.

Focus of $y^2 = 16x$ is $(4,0)$.

Any line through this point will be $mx - y - 4m = 0$.

As this is tangent to the circle hence $\left|\frac{6m-0-4m}{\sqrt{m^2+1}}\right| = \sqrt{2}$.

$\Rightarrow m = 1.$