Solutions

PARABOLA

Ex. 2(B)

Q.1 (C), (D)

Coordinates of extremes of latus rectum of $y^2 = 4ax$ are $(a, \pm 2a)$, hence

for the parabola $(y-k)^2 = 4a(x-h)$ the coordinates will be $(h+a, k \pm 2a)$.

Now for $(y-1)^2 = 2(x+2)$ we get $\left(-2+\frac{1}{2}, 1\pm 1\right)$ i.e. $\left(-\frac{3}{2}, 2\right) \& \left(-\frac{3}{2}, 0\right)$.

Q.2 (A), (C)

Any tangent to $y^2 = 8x$ will be $y = mx + \frac{2}{m}$.

Now slope of the given line is 3 and angle between this line and tangent is 45^o.

$\Rightarrow \tan 45^o = \pm \frac{m-3}{1+3m}$ or $m = -2$ & $m = \frac{1}{2}$.

Required tangents are $2x + y + 1 = 0$ & $2y = x + 8$.

Q.3 (A), (C)

Let mid-point of chord be (h, k), then by $T = S_1$, eq. of chord will be

$hx - 2(y+k) = h^2 - 4k$. As it passes through $(2,1)$ or $(-2,1)$, hence

$2h - 2(1+k) = h^2 - 4k$ or $-2h - 2(1+k) = h^2 - 4k$.

Required locus is $x^2 = 2(x+y-1)$ or $x^2 = 2(-x+y-1)$.

Q.4 (A), (B)

If a chord joining $P(t_1) \& Q(t_2)$ subtends a right angle at vertex, then $t_1 t_2 = -4$.

Similarly if PQ is a normal chord, then $t_2 = -t_1 - \frac{2}{t_1}$.

From these two relations we get $-\frac{4}{t_1} = -t_1 - \frac{2}{t_1}$ or $t_1 = \pm\sqrt{2}$ & $t_2 = \mp 2\sqrt{2}$.

Now slope of $PQ = \frac{2}{t_1 + t_2}$ or $\pm\sqrt{2}$.

Q.5 (A), (B), (C), (D)

All standard geometrical results/properties of a parabola.

Q.6 (B), (D)

$y^2+2ax+2by+c=0 \Rightarrow (y+b)^2=-2a\left(x-\frac{b^2-c}{2a}\right).$

Hence locus is a parabola with latus rectum 2a.

Q.7 (A), (B)

Let the tangent be $ty=x+\frac{9}{4}t^2.$

As it passes through $(4,10)$ hence $9t^2-40t+16=0 \Rightarrow t=\frac{4}{9}\ \&\ 4.$

Points of contact are $\left(\frac{4}{9},2\right)\&(36,18).$

Q.8 (A), (B), (C)

Any circle passing through $P\left(\frac{1}{2},1\right)$ & $V(0,0)$ will be

$\left(x-\frac{1}{2}\right)x+(y-1)y+\lambda(y-2x)=0$ or $x^2+y^2-\left(2\lambda+\frac{1}{2}\right)x+(\lambda-1)y=0.$

Now normal to $y^2=2x$ at $\left(\frac{1}{2},1\right)$ is $2x+2y-3=0$

As the circle touches the parabola hence the normal will pass through center of the circle.

$\Rightarrow 2\left(\lambda+\frac{1}{4}\right)-2\frac{(\lambda-1)}{2}-3=0$ or $\lambda=\frac{3}{2}.$

Hence equation of circle is $x^2+y^2-\frac{7}{2}x+\frac{1}{2}y=0.$

Q.9 (A), (B)

Equation of $AB: 2y=2(x-1)$ or $y=x-1.$ $\{\text{by } T=0\}$

Now as $(-1,2)$ lies on directrics hence AB must pass through the focus $S(1,0)$.

Any point on AB at a distance r from S will be $\left(1+\frac{r}{\sqrt{2}},\frac{r}{\sqrt{2}}\right).$

Substituting these coordinates in $y^2=4x$ gives $r^2-4\sqrt{2}r-8=0.$

Now SA & $-$SB will be roots of this equation hence $AB=SA+SB.$

$SA+(-SB)=4\sqrt{2}$ & $SA\times SB=8$ & $SA+SB=\sqrt{(SA-SB)^2+4SA\times SB}$

$\Rightarrow SA+SB=8.$

Q.10 (B), (C)

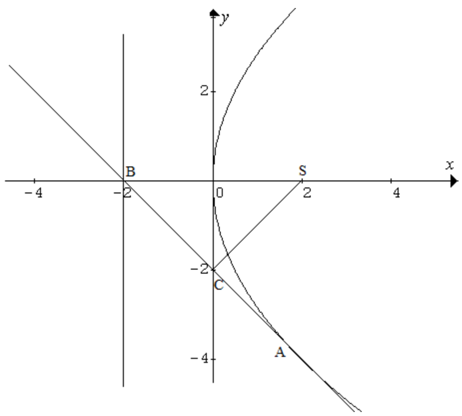
If a chord joining $P(t_1) \& Q(t_2)$ subtends a right angle at vertex, then $t_1t_2 = -4.$

Similarly if PQ is a normal chord, then $t_2 = -t_1 - \dfrac{2}{t_1}.$

From these two relations we get $-\dfrac{4}{t_1} = -t_1 - \dfrac{2}{t_1}$ or $t_1 = \pm\sqrt{2}.$

Hence coordinates of P will be $\left(2a, \pm 2\sqrt{2}\,a\right).$

Q.11 (A), (B), (C)



Option (A) is geometrical property of a parabola.
By comparing $ty = x + 2t^2$ with $-y = x + 2$ we get point of contact as (2, –2).
Now refer the adjoining figure.

Q.12 (A), (B), (C), (D)

Standard geometrical properties of a parabola.

Q.13 (C), (D)

If the circle cuts parabola orthogonally, then tangent to the parabola at the point of intersection of the two curves will pass through the center of the circle.
Now any tangent to the parabola will be $ty = x + t^2$. If it passes through (2, 3), then
$t^2 - 3t + 2 = 0$ or $t = 1$ & $t = 2$.
Now the points are (1, 2) & (4, 4).

Q.14 (B), (C)

Tangent to parabola : $y = mx + \dfrac{1}{2m}$

Tangent to circle : $y = m(x-4) \pm 4\sqrt{1+m^2}$.

For common tangent, $\dfrac{1}{2m} = -4m \pm 4\sqrt{1+m^2} \Rightarrow m = \pm\dfrac{1}{4\sqrt{3}}$.

Q.15 (A), (C)

Focal distance of any point P(t) on $y^2 = 8x$ is $2 + 2t^2$.

Now $2 + 2t^2 = 4 \Rightarrow t = \pm 1$.

Required points are $(2, 4) \& (2, -4)$.

Q.16 (B), (C)

Refer Q.4 of this exercise.

Q.17 (A), (D)

Any normal to $y^2 = 4ax$ will be $tx + y = 2at + at^3$.

If this normal passes through (5a, 2a), then $t^3 - 3t - 2 = 0$.

Now the above equation has two real roots i.e. t = –1 & 2.

Hence two normal from (5a, 2a) are y = x – 3a & y + 2x = 12a.

Q.18 (A), (B)

Standard geometrical properties of a parabola.

Q.19 (B), (C)

The eq. $(x-1)^2 + (y-1)^2 = \left(\frac{x+y}{\sqrt{2}}\right)^2$ represents a parabola having focus at S(1, 1) & directrics as x + y = 0.

Now axis will be line perpendicular to the directrics and passing through focus i.e. x – y = 0.

Axis meets the directrics at Z(0, 0) hence vertex will be the midpoint of S & Z i.e. $V\left(\frac{1}{2}, \frac{1}{2}\right)$.

Also focal distance of the vertex is VS = $\frac{1}{\sqrt{2}}$.

By properties of a parabola the point P on axis at a distance VS from the focus is the point from which 2 coincident and one distinct normal (axis itself) can be drawn to the parabola and from any point on axis beyond this point (away from focus) three normal may be drawn.

Such point P on axis will be $(1 + VS\cos\theta, 1 + VS\sin\theta)$ i.e. $\left(\frac{3}{2}, \frac{3}{2}\right)$, where $\tan\theta = 1$ is slope of axis.

Hence for any point from which three normal can be drawn h > 3/2, k > 3/2.

Q.20 (A), (B)

$f(x+y) = f(x)f(y)$ & $f(1) = 2 \Rightarrow f(x) = 2^x$.

Now focal distance of any point $(2t^2, 4t)$ on the parabola $y^2 = 8x$ will be $2 + 2t^2$.

$2 + 2t^2 = 4 \Rightarrow t = \pm 1$

Required point may be (2, 4) or (2, –4), but $a_r = 2^r$ gives $a_1 = 2$ & $a_2 = 4$.

Hence the points are $(a_1, \pm a_2)$.

PASSAGE I

Q.21 (D) Q.22 (A) Q.23 (A)

Differentiating the equations of the parabolas gives $y' = \dfrac{2a}{y}$ & $y' = \dfrac{x}{2a}$.

As the parabolas touch hence at the point of contact there gradients must be equal.

$\dfrac{2a}{y} = \dfrac{x}{2a} \Rightarrow (x, y)$ lies on $xy = 4a^2$.

Also projections of (x, y) on coordinate axes will be (x, 0) & (0, y).

Hence area of rectangle OQPR will be (x.y) i.e. $4a^2$.

Perimeter of the rectangle will be 2(|x| + |y|), which will be minimum when x = y.

Hence least perimeter is 8a.

PASSAGE II

Q.24 (A) Q.25 (B) Q.26 (B)

$$\text{Area} = \frac{1}{2}\left\|\begin{matrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{matrix}\right\| = \frac{|a-b||b-c||c-a|}{2}.$$

Clearly area is an integer.

Length of AB = $\sqrt{(a-b)^2 + (a^2-b^2)^2} = |a-b|\sqrt{1+(a+b)^2}$.

As '1' does not appear in any Pythagorean triplet of integers hence length is irrational.

As area in integer and side length is irrational, hence length of altitude must be irrational.

PASSAGE III

Q.27 (C) Q.28 (A) Q.29 (B)

By given information length of Sub tangent = twice the abscissa i.e. 8.

For any point (h, k), S. T. = S. N. $\Rightarrow$ 2h = 2a or h = a & k = $\pm 2a$.

At any point (h, k), S.T. = 2h, Ordinate = k & S.N. = 2a, where $k^2 = 4ah$ implies these are in G.P.

PASSAGE IV

Q.30 (A) Q.31 (C) Q.32 (B)

Slope of tangent at P(t) = $\frac{1}{t}$. Let slope of any line making an angle α with this tangent be m,

then $\frac{tm-1}{t+m} = \pm \tan\alpha$ or $m = \frac{1 \pm t\tan\alpha}{t \mp \tan\alpha}$.

Now if this line joins P(t) & Q(t_1), then its slope must be $\frac{2}{t+t_1}$.

Hence for Q(t_1) & Q'(t_2) we get $\frac{t+t_1}{2} = \frac{t+\tan\alpha}{1-t\tan\alpha}$ & $\frac{t+t_2}{2} = \frac{t-\tan\alpha}{1+t\tan\alpha}$

Which implies $t_1 = \frac{t+\left(t^2+2\right)\tan\alpha}{1-t\tan\alpha}$ & $t_2 = \frac{t-\left(t^2+2\right)\tan\alpha}{1+t\tan\alpha}$

or $t_1+t_2 = \frac{2t+2t\left(t^2+2\right)\tan^2\alpha}{1-t^2\tan^2\alpha}$ & $t_1t_2 = \frac{t^2-\left(t^2+2\right)^2\tan^2\alpha}{1-t^2\tan^2\alpha}$.

Also equation of QQ' is $\left(t^2\tan^2\alpha-1\right)x+\left(\left(t^2+2\right)\tan^2\alpha+1\right)ty+a\left(\left(t^2+2\right)^2\tan^2\alpha-t^2\right)=0$.

Eq. of QQ' can be rearranged as $\left(t^2x+\left(t^2+2\right)ty+a\left(t^2+2\right)^2\right)\tan^2\alpha-\left(x-ty+at^2\right)=0$, hence this line will always pass through the point of intersection of the lines

$t^2x+\left(t^2+2\right)ty+a\left(t^2+2\right)^2=0$ & $x-ty+at^2=0$ i.e. $\left(-2a-at^2,-\frac{2a}{t}\right)$.

Q. 33 (A)

Both the statements are standard results in context of a parabola.

Q.34 (A)

Both the statements are standard results in context of a parabola.

Q.35 (C)

Given equation can be rewritten as $\left(y+\frac{2}{3}\right)^2 = 2\left(x-\frac{10}{9}\right)$.

Now in $y^2 = 4ax$ we can draw three normal from any point (h, k) if h > 2a.

Here a is ½ and h has to be replaced by h – 10/9, hence to draw three normal x > 19/9.

Q.36 (D)

Statement (R) is a standard property of A.P. & G.P. that if a, b, c are in A.P. as well as G.P., then a = b = c, but for the parabola $x^2 = 4ay$, if the ordinates $at_1^2 = at_2^2 = at_3^2$, then it's not necessary that $t_1 = t_2 = t_3$ so the points are not necessarily same.

Point of intersection of tangents at $P(t_1) \& R(t_2)$ is given by $(at_1t_2, a(t_1+t_2))$.

Now for $Q(t_3)$ it is given that $2at_1, 2at_2 \& 2at_3$ are in G.P. i.e. $t_2^2 = t_1t_3$.

A point on any line through Q, parallel to axis, will be having ordinate as $2at_3$.

But $a(t_1+t_2) \neq 2at_3$.

Q.37 (C)

Statement (R) is clearly false.

Now any tangent to the parabola $y^2 = 9x$ will be $y = mx + \frac{9}{4m}$.

As it passes through (4, 10), hence $10 = 4m + \frac{9}{4m}$ or $16m^2 - 40m + 9 = 0 \Rightarrow m = \frac{1}{4}, \frac{9}{4}$.

Q.38 $(\mathbf{A}) \to (\mathbf{Q}), (\mathbf{B}) \to (\mathbf{R}), (\mathbf{C}) \to (\mathbf{S})$

(A) Slope of the given line is –1. Now minimum distance between this line and the parabola will be measured along common normal to the curve and the line.

Eq. of normal to $y^2 = 4(x-9)$ with slope m will be given by $y = m(x-9) - 2m - am^3$.

For m = 1, we get x – y = 12.

Now this line will meet x + y = 4 at the point (8, – 4) $\therefore \left|\frac{p_1}{p_2}\right| = 2$.

(B) Slope of normal to $y^2 = 4x$ at $(t^2, 2t)$ is $-t$, hence for $A(t_1), B(t_2) \& C(t_3)$ we have $t_1 = -2, t_2 = -1 \& t_3 = 3$.

Now $t_1t_2 + t_2t_3 + t_3t_1 = 2 - h \Rightarrow h = -7$ & $t_1t_2t_3 = k \Rightarrow k = 6$.

Hence point of concurrency is (–7, 6) and FP = 10

Focal distance of P(t) will be $1 + t^2$, hence FA = 5, FB = 2 & FC = 10.

Therefore $\frac{FA \times FB \times FC}{FP^2} = 1$.

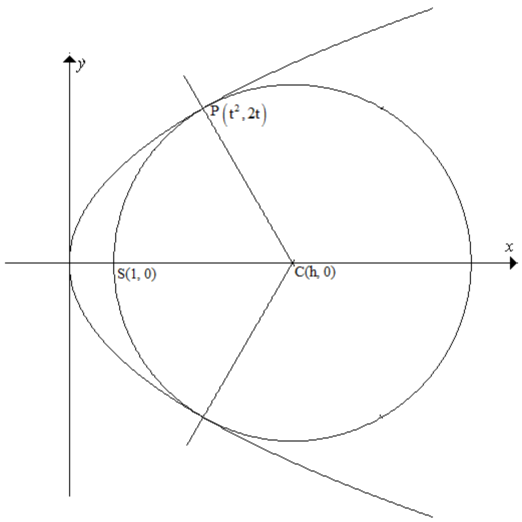
(C) As normal at $B(t_1)$ & normal at $C(t_2)$ intersect at A(t) hence $t_1t_2 = 2$.

Now equation of chord BC will be $(t_1+t_2)y = 2x + 4t_1t_2$ or $y = \frac{2}{t_1+t_2}(x+4)$.

Hence the fixed point M is (–4, 0). Distance of M from the origin = 4.

Q.39 $(A)\to(S),(B)\to(Q),(C)\to(R),(D)\to(P)$

(A) Clearly such a circle will have its diameter along the axis i.e. x – axis. Let the center of required circle be C(h, 0) and radius be $r=|h-1|$.



Also (h, 0) will lie on some normal to the parabola as the largest such circle will touch the parabola.

Substituting (h, 0) in equation of normal at $P(t^2,2t)$ gives $t^2=h-2$.

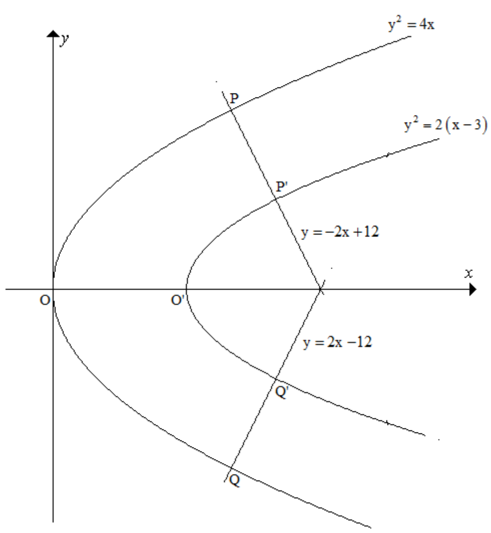
Now for this circle to be contained inside the parabola, distance of C from the point $P(t^2,2t)$ on the parabola must be greater than or equal to the radius of the circle.

Now $CP\ge|h-1|\Rightarrow(h-t^2)^2+4t^2\ge(h-1)^2$.

For $t^2=h-2$ we get $(h-1)^2\le4(h-1)$ or $h\le5$.

hence largest possible radius is 4.

(B)



Any normal to $y^2=4x$ is $y=mx-2m-m^3$ and any normal to $y^2=2(x-3)$ is $y=m(x-3)-m-\frac{1}{2}m^3$.

Comparing the two equations gives

$-2m-m^3=-4m-\frac{1}{2}m^3$ or $m=-2,0,2$.

hence the common normal to the two parabolas are x–axis, $y=2x-12$ & $y=-2x+12$.

Now point of contact of normal for $y^2=4x$ is $(m^2,-2m)$, hence P is (4, 4).

Similarly point of contact of normal for $y^2=2(x-3)$ is $\left(\frac{m^2}{2}+3,-m\right)$, hence P' is (5, 2).

Now OO' = 3, PP' = $\sqrt{5}$. Therefore least distance is $\sqrt{5}$ and d^2 = 5.

(C) Harmonic mean of segments of Focal chord = Semi Latus rectum.

(D) AB must be a focal chord and shortest focal chord is Latus rectum.