Solutions
PARABOLA
Ex.2(A)

Q.1

$$25\left(x^2+y^2-2x+1\right)=(4x-3y+1)^2 \Rightarrow \sqrt{(x-1)^2+y^2}=\left(\frac{4x-3y+1}{5}\right)^2$$

Hence (x,y) moves such that its distance from $(1,0)$ is equal to its distance from $4x-3y+1=0$, which means locus will be a parabola with focus at $(1,0)$ & directrics as $4x-3y+1=0$.

Q.2

(i) Let P be $\left(at_1^2,\ 2at_1\right)$ & Q be $\left(at_2^2, 2at_2\right)$, then circle on PQ as diameter will be

$$\left(x-at_1^2\right)\left(x-at_2^2\right)+\left(y-2at_1\right)\left(y-2at_2\right)=0$$

or $x^2+y^2-a\left(t_1^2+t_2^2\right)x-2a\left(t_1+t_2\right)y-3a^2=0$

As tangents at the ends of any focal chord are mutually perpendicular & intersect on the directrics hence his circle passes through $(-a,k)$ on the directrix. Substituting $(-a,k)$ in the equation of circle gives $a^2+k^2+a^2\left(t_1^2+t_2^2\right)-2a\left(t_1+t_2\right)k-3a^2=0$

or $k^2-2a\left(t_1+t_2\right)k+a^2\left(t_1+t_2\right)^2-2a^2t_1t_2-2a^2=0$

Now as PQ is a focal chord, then $t_1t_2=-1$, hence $k^2-2a\left(t_1+t_2\right)k+a^2\left(t_1+t_2\right)^2=0$

Or $\left(k-a\left(t_1+t_2\right)\right)^2=0$. Clearly the circle touches the directrics at $\left(-a,a\left(t_1+t_2\right)\right)$.

(ii) Let P be $\left(at^2,\ 2at\right)$ & F be $(a,0)$, then circle on PF as diameter will be

$\left(x-at^2\right)(x-a)+\left(y-2at_1\right)y=0$ or $x^2+y^2-a\left(t^2+1\right)x-2aty+a^2t^2=0$

As foot of perpendicular from focus on any tangent lies on the tangent at vertex hence his circle passes through $(0,k)$ on the $y-$axis. Substituting $(0,k)$ in the equation of circle gives $k^2-2akt+a^2t^2=0$

Or $(k-at)^2=0$. Clearly the circle touches the $y-$axis at $(0,at)$.

Q.3

Latus rectum = 2 × distance between focus & foot of directrics.

Hence $\text{L.R.}=2\sqrt{(1+3)^2+(1-4)^2}=10$.

Q.4

Projected path is a parabola whose Vertex: $\left(\frac{1}{2},4\right)$, Axis: parallel to y-axis

Equation of the parabola will be $\left(x-\frac{1}{2}\right)^2=-4a(y-4)$.

As it passes through $(0,0)$ hence $a=\frac{1}{64}$.

Now for $x=\frac{3}{4}$, $\left(\frac{3}{4}-\frac{1}{2}\right)^2=-\frac{1}{16}(y-4)\Rightarrow y=3$ m.

Q.5

Given $S:(a,0), X:(-a,0)$. Let P be $(at^2,2at)$ & P' be $(at^2,-2at)$.

$PX:\frac{y}{x+a}=\frac{2at}{at^2+a}$. Let Q be $(at_1{}^2,2at_1)$, then $\frac{2at_1}{at_1{}^2+a}=\frac{2at}{at^2+a}$

or $t^2t_1+t_1=t_1{}^2t+t\Rightarrow t_1=\frac{1}{t}$, hence Q is $\left(\frac{a}{t^2},\frac{2a}{t}\right)$.

Now $P'Q:\frac{y-2at}{x-at^2}=\frac{2at-\frac{2a}{t}}{at^2-\frac{a}{t^2}}$ or $\frac{y-2at}{x-at^2}=\frac{2t}{t^2+1}$.

Clearly $(a,0)$ satisfies the eq. of P'Q.

Q.6

Let equation of tangent be $x-ty=at^2$.
Any point on a line of slope $\tan\theta$ and passing through origin will be $X(r\cos\theta,r\sin\theta)$, where r is distance of X from O.
Hence P will be $(OP\cos\theta,OP\sin\theta)$ & Q will be $(OQ\cos\theta,OQ\sin\theta)$.

Substituting P in $x-ty+at^2=0$ gives $OP=\left|\frac{at^2}{t\sin\theta-\cos\theta}\right|$.

Substituting Q in $y^2=4ax$ gives $OQ=\left|\frac{4a\cos\theta}{\sin^2\theta}\right|$.

Not $\tan\theta=-t\Rightarrow\cos\theta=-\frac{1}{\sqrt{1+t^2}}$ & $\sin\theta=\frac{t}{\sqrt{1+t^2}}$.

Hence $OP\times OQ=\left|\frac{at^2}{\frac{t^2}{\sqrt{1+t^2}}+\frac{1}{\sqrt{1+t^2}}}\right|\times\left|\frac{4a\frac{1}{\sqrt{1+t^2}}}{\frac{t^2}{1+t^2}}\right|=4a^2$.

Q.7

Equation of tangent of slope m to $y^2 = 16x$ is $y = mx + \frac{4}{m}$.

Slope of the given line $= 2$, hence slopes of required tangents will be 2 & $-\frac{1}{2}$.

Thus required tangents are $2x - y + 2 = 0$ & $x + 2y + 16 = 0$.

Q.8

Coordinates $R : (at_1t_2, a(t_1 + t_2))$.
As normals at P & Q meet on the curve hence $t_1t_2 = 2$.
Required locus is $x = 2a$.

Q.9

Let the tangents be $t_1y = x + 2t_1^2$ & $t_2y = x + 2t_2^2$.
These lines will meet $y-$axis in $P(0, 2t_1)$ & $Q(0, 2t_2)$ hence $PQ = 2|t_1 - t_2| = 4$.
Now point of intersection of tangents will be $x = 2t_1t_2, y = 2(t_1 + t_2)$
Now $y^2 = 4(t_1 + t_2)^2$ or $y^2 = 4(t_1 - t_2)^2 + 16t_1t_2$
$\therefore$ Required locus is $y^2 = 8(x + 2)$.

Q.10

Let mid point of any chord of $x^2 + y^2 = r^2$ be (h, k), then equation will be $\{\text{By } T = S_1\}$
$hx + ky = h^2 + k^2$ or $y = -\frac{h}{k}x + \frac{h^2 + k^2}{k}$.
Here $m = -\frac{h}{k}$ & $c = \frac{h^2 + k^2}{k}$ and for this line being a tangent to $y^2 = 4ax$, $cm = a$.
$\therefore \frac{h^2 + k^2}{k} \times \frac{h}{k} = -a \Rightarrow h(h^2 + k^2) = -ak^2$.
Required locus is $x(x^2 + y^2) + ay^2 = 0$.

Q.11

Any circle touching the $x-$axis may be given by $(x - h)^2 + (y - k)^2 = k^2$.
Normal to $y^2 = 4ax$ at one end of latus rectum i.e. $(a, 2a)$ is $x + y = 3a$
As it is touching the parabola at $(a, 2a)$ hence normal to parabola at this point will pass through (h, k) & circle will pass through this point.
$\Rightarrow h + k = 3a$ & $(a - h)^2 + (2a - k)^2 = k^2$ or $(3a - k)^2 - 2a(3a - k) - 4ak + 5a^2 = 0$
$\Rightarrow k^2 - 8ak + 8a^2 = 0$ or $k = 4 \pm 2\sqrt{2}$. Hence radius is $4 + 2\sqrt{2}$ or $4 - 2\sqrt{2}$.

Q.12

By linearly combining eqs. of two intersecting curves we get eq. of a curve passing through their point of intersections.

Now by adding $y^2 = 4x$ & $x^2 = 4y$ we will get $x^2 + y^2 - 4x - 4y = 0$.

This is the required circle.

Q.13

Tangent at $P\left(at^2, 2at\right)$ be $ty = x + at^2$. Also the focus is $F\left(a, 0\right)$ directrics is $x = -a$.

Hence Q will be $\left(-a, \frac{at^2 - a}{t}\right)$. Now Slope of $PF = \frac{2at}{at^2 - a}$ & that of $QF = \frac{at^2 - a}{-2at}$.

Clearly product of slopes is -1, hence $\angle PFQ = \frac{\pi}{2}$.

Q.14

Focus of $x^2 = 4y$ is $(0,1)$.

Now eq. of tangent to the parabola at $(6,9)$ is $3x - y = 9$. $\{\text{By } T = 0\}$

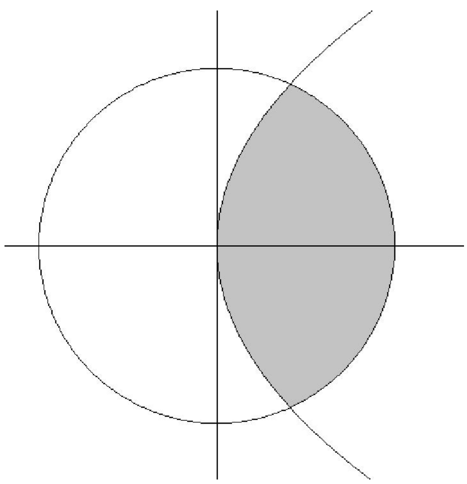
Eq. of any circle touching $3x - y = 9$ at $(6,9)$ may be represented as family of a circle of zero radius and center at $(6,9)$ & the line $3x - y = 9$ i.e.

$(x-6)^2 + (y-9)^2 + \lambda(3x - y - 9) = 0$.

As it passes through $(0,1)$, hence $\lambda = 10$.

Required circle is $x^2 + y^2 + 18x - 28y + 27 = 0$.

Q.15



Point $(-2\alpha, \alpha + 1)$ must lie inside both the curves in the shaded region shown in figure.

Hence $(-2\alpha)^2 + (\alpha + 1)^2 < 4$ & $(\alpha + 1)^2 < -8\alpha$.

$\Rightarrow 5\alpha^2 + 2\alpha - 3 < 0$ & $\alpha^2 + 10\alpha + 1 < 0$.

$\Rightarrow -1 < \alpha < \frac{3}{5}$ & $-5 - 2\sqrt{6} < \alpha < -5 + 2\sqrt{6}$.

Taking intersection of the two intervals gives $-1 < \alpha < -5 + 2\sqrt{6}$.

Q.16

Any tangent to $y^2 = 4a(x+b)$ will be given by $y = m'(x+b) + \frac{a}{m'}$ or $y = m'x + bm' + \frac{a}{m'}$.

Given line is $y = -\frac{m}{\ell}x + m$.

Comparing the two equations gives $m' = -\frac{m}{\ell}$ & $m = bm' + \frac{a}{m'}$.

Eliminating m' gives $m = -\frac{bm}{\ell} - \frac{al}{m}$ or $m^2(\ell + b) + a\ell^2 = 0$.

Q.17

Any tangent to $y^2 = 4x : y = mx + \frac{1}{m}$.

Any tangent to $(x-3)^2 + y^2 = 9 : y = m(x-3) \pm 3\sqrt{1+m^2}$.

For common tangent $\frac{1}{m} = -3m \pm 3\sqrt{1+m^2}$.

$\Rightarrow \left(\frac{1}{m} + 3m\right)^2 = 9(1+m^2)$ or $\frac{1}{m^2} + 9m^2 + 6 = 9 + 9m^2$.

$\Rightarrow m = \pm\frac{1}{\sqrt{3}}$.

Also both the curves are touching $x-$axis at $(0,0)$.

All possible common tangents are $x = 0, \pm\sqrt{3}y = x + 3$.

Q.18

Any tangent of slope m to $y^2 = 4(x-9)$ will be $y = m(x-9) + \frac{1}{m}$

Slope of the given line is -1.

Equation of tangent parallel to the given line will be $x + y = 8$.

Point corresponding to this tangent on the parabola is $(8,-4)$.

This will be the nearest point on the parabola.

Q.19

Let coordinates of these points be $P(at_1^2, 2at_1), Q(at_2^2, 2at_2)$ & $R(at_3^2, 2at_3)$.

Given $2at_1, 2at_2, 2at_3$ are in G.P. hence $t_2^2 = t_1t_3$.

Now point of intersection of tangents at P & R will be $(at_1t_3, a(t_1+t_3))$ or $(at_2^2, a(t_1+t_3))$.

Clearly this point lies on a line passing through Q and parallel to $y-$axis.

Q.20

Any tangent to $y^2 = 8ax : y = mx + \frac{2a}{m}$.

Any tangent to $x^2 + y^2 = 2a^2 : y = mx \pm a\sqrt{2}\sqrt{1+m^2}$.

For common tangent $\frac{2a}{m} = \pm a\sqrt{2}\sqrt{1+m^2}$.

$\Rightarrow \frac{4a^2}{m^2} = 2a^2\left(1+m^2\right)$ or $m^4 + m^2 - 2 = 0 \Rightarrow m = \pm 1$.

All possible common tangents are $\pm y = x + 2a$.

Q.21

Length of focal chord with $P\left(t_1\right) \& Q\left(t_2\right)$ as end points $= a\left(t_1 - t_2\right)^2$.

Also slope of PQ, $\frac{2}{t_1 + t_2} = \tan\theta$.

Now $t_1 + t_2 = 2\cot\theta \Rightarrow \left(t_1 - t_2\right)^2 = \left(t_1 + t_2\right)^2 - 4t_1t_2$, but for focal chord $t_1t_2 = -1$.

Hence $\left(t_1 - t_2\right)^2 = 4\operatorname{cosec}^2\theta$.

Length of focal chord $= 4a\operatorname{cosec}^2\theta$.

Q.22

Equation of normal to $y^2 = 4x$ in parametric form : $tx + y = 2t + t^3$.

As it is drawn through $(15,12)$ therefor $t^3 - 13t - 12 = 0$.

Or $(t+1)\left(t^2 - t - 12\right) = 0$. Hence $t = -1, 4, -3$.

Normals are $x - y = 3,\ 4x + y = 72,\ 3x - y = 33$.

Q.23

Any point on $y^2 = 8x$ will be $\left(2t^2, 4t\right)$. $P(18,12) \equiv \left(2t^2, 4t\right) \Rightarrow t = 3$.

Now slope of normal at P will be $\tan\theta = -t = -3 \Rightarrow \cos\theta = -\frac{1}{\sqrt{10}} \& \sin\theta = \frac{3}{\sqrt{10}}$.

A point on this normal at a distance PQ from P will be $\left(18 - \frac{PQ}{\sqrt{10}}, 12 + \frac{3PQ}{\sqrt{10}}\right)$.

Substituting these coordinates in the equation of parabola gives $\left(12 + \frac{3PQ}{\sqrt{10}}\right)^2 = 8\left(18 - \frac{PQ}{\sqrt{10}}\right)$

Hence $9PQ = 80\sqrt{10}$.

Q.24

Circle drawn through T, P & Q will also pass through the point of intersection of normals at P & Q and will have TN as diameter.

Now point of intersection of tangents at P(t_1) & Q(t_2) is $T\left(2t_1t_2, 2\left(t_1+t_2\right)\right)$

Comparing with (12, 10) gives $t_1 = 2$ & $t_3 = 3$

Hence point of intersection of normals $N\left(2\left(t_1^2+t_1t_2+t_2^2+2\right), -2t_1t_2\left(t_1+t_2\right)\right)$ will be $(42, -60)$.

Mid point of TN is (27, –25).

Q.25

Let the normal chord be the line joining P(t_1) and Q(t_2) where $t_2 = -t_1 - \frac{2}{t_1}$.

Now mid point of PQ will be $x = t_1 + t_2, y = \frac{t_1^2 + t_2^2}{2}$.

$$t_2 = -t_1 - \frac{2}{t_1} \Rightarrow x = t_2 + t_1 = -\frac{2}{t_1} \ \& \ 2y = \left(t_1 + t_2\right)^2 - 2t_1t_2 = \frac{4}{t_1^2} + 2\left(t_1^2 + 2\right)$$

Eliminating t_1 between x & y gives $2y = x^2 + \frac{8}{x^2} + 4$.

Q.26 (A)

Point of intersection of tangents at $\left(at_1^2, 2at_1\right)$ & $\left(at_2^2, 2at_2\right)$ is $\left(at_1t_2, a\left(t_1+t_2\right)\right)$.

Clearly y_1, y_3, y_2 are in A.P.

Q.27 (C)

$x^2 + y^2 - 2gx = 0 \Rightarrow (x - 0)(x - g) + (y - 0)(y - 0) = 0$.

Circle is drawn with (g, 0) & (0, 0) as extremities of diameter and as $g < 0$ hence circle is touching the y – axis at origin from left hand side.

Also $y^2 = 4ax$ $(a > 0)$ is touching the y – axis at origin from right hand side of y – axis having x – axis as axis.

The two curves touch each other at the origin.

Q.28 (C)

$$\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a} \Rightarrow \frac{1}{6} + \frac{1}{SQ} = \frac{1}{2} \text{ or } SQ = 3.$$

Q.29 (B)

Such a tangent can be drawn only at vertex, hence P must be the vertex i.e. (–2, 4).

Q.30 (B)

$$x^2 = 2(1-\cos t) \Rightarrow x^2 = 4\sin^2\frac{t}{2} \text{ or } x^2 = 4 - y.$$

Q.31 (D)

Any tangent to $y^2 = 4ax$ is $y = mx + \frac{a}{m}$.

As it is normal to $x^2 + y^2 - 2ax - 2by + c = 0$ hence $y = mx + \frac{a}{m}$ must pass through (a, b).

$$\therefore\ b = ma + \frac{a}{m} \text{ or } am^2 - bm + a = 0.$$

For two such tangents, this equation must have real & distinct roots.

$\Rightarrow b^2 > 4a^2$.

Q.32 (A)

The point P must be on same side as the origin of line drawn through A & B i.e. $x + y = 4$.

Hence $p^2 + p - 6 < 0$ ...(i)

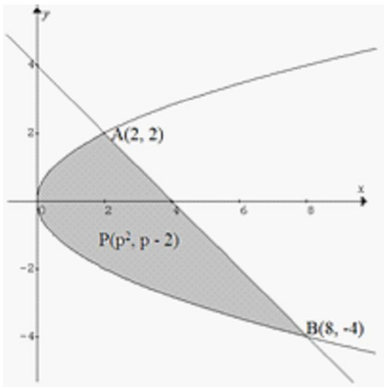
$\Rightarrow -3 < p < 2$.

Also P lies inside the parabola hence $(p-2)^2 < 2p^2$

or $p^2 + 4p - 4 > 0$.

$\Rightarrow p < -2 - 2\sqrt{2}$ or $p > -2 + 2\sqrt{2}$ ...(ii)

From (i) & (ii) $-2 + 2\sqrt{2} < p < 2$.



Q.33 (B)

Center (6, 0) of the circle lies inside the parabola, hence if the circle is also contained inside the parabola there will not be any common tangent.

Hence radius of the circle must be less than the minimum distance of (6, 0) from any point of the parabola.

Now any normal at $(m^2, -2m)$ to the parabola is $y = mx - 2m - m^3$.

Let this normal pass through (6, 0), then $m^3 = 4m$ or $m = 0, 2, -2$.

Feet of normals are (0, 0), (4, –4) & (4, 4).

Least distance of (6, 0) from the parabola is $\sqrt{20}$ hence $r < \sqrt{20}$.

Q.34 (B)

Any tangent to $y^2 = 4ax$ is $y = mx + \frac{a}{m}$ &

Any normal to $x^2 = 4by$ is $y = mx + 2b + \frac{b}{m^2}$.

From these two equations we get $2bm^2 - am + b = 0$.

For two such lines, $a^2 > 8b^2$ or $|a| > 2\sqrt{2}|b|$.

Q.35 (A)

Standard result : If $y^2 = 4ax$ meets a circle in 4 points $(at_i^2, 2at_i)$, i = 1, 2, 3, 4, then

$t_1 + t_2 + t_3 + t_4 = 0$. (to get this result put $(at^2, 2at)$ in the equation of the circle and form a 4^{th} degree equation in t whose roots will be t_1, t_2, t_3, t_4)

Hence $y_1 + y_2 + y_3 + y_4 = 0$.

Q.36 (C)

Standard result : Any tangent to $y^2 = 4a(x + a)$ is $y = m(x + a) + \frac{a}{m}$.

Q.37 (A)

Standard fact : Harmonic mean of segments of any focal chord is semi Latus rectum.

Q.38 (A)

Standard result : length of focal chord having on extremity as $(at^2, 2at)$ is $a\left(t + \frac{1}{t}\right)^2$.

Q.39 (C)

Equation of tangent to $x^2 = y$ at (2, 4) is $4x - y = 4$.

Any circle touching $4x - y - 4 = 0$ at (2, 4) will be $(x - 2)^2 + (y - 4)^2 + \lambda(4x - y - 4) = 0$.

As this circle also passes through (0, 1) hence $\lambda = 13/5$.

$\therefore$ Equation of the circle : $x^2 + y^2 + \frac{32}{5}x - \frac{53}{5}y + \frac{48}{5} = 0$ and enter of the circle : $\left(-\frac{16}{5}, \frac{53}{10}\right)$.

Q.40 (A)

Let midpoint of the chord be $(at^2, 2at)$, then equation of the chord will be

$at^2x + 2aty = a^2t^4 + 4a^2t^2$.

As it is drawn from (a, a) hence $t(t^3 + 3t - 2) = 0$.

Now $t^3 + 3t - 2 = 0$ has no real roots so only 1 such chord is possible.